On the hierarchy of linearity axioms with respect to logic and arithmetic*

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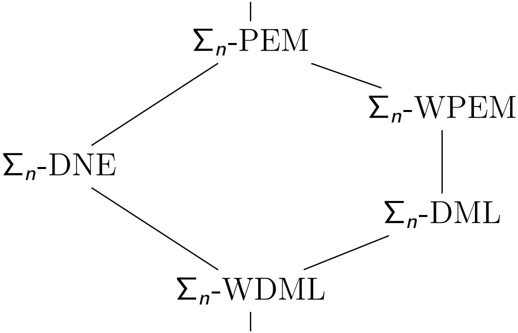
Logical Principles

The following principles are classically valid but not provable in intuitionistic theories:

PEM (the principle of excluded middle) : $\varphi \vee \neg\varphi$;
DML (the De Morgan law) : $\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$;
DNE (the double negation elimination) : $\neg\neg\varphi \to \varphi$;

WPEM (the weak principle of excluded middle) : $\neg\varphi \vee \neg\neg\varphi$;
WDML (the weak De Morgan law) : $\neg(\neg\varphi \wedge \neg\psi) \to \neg\neg\varphi \vee \neg\neg\psi$.

Arithmetical hierarchy over HA + Σ_{n-1}-DNE



$$\Sigma_1\text{-PEM} : \exists x A_0(x) \vee \neg\exists x A_0(x);$$

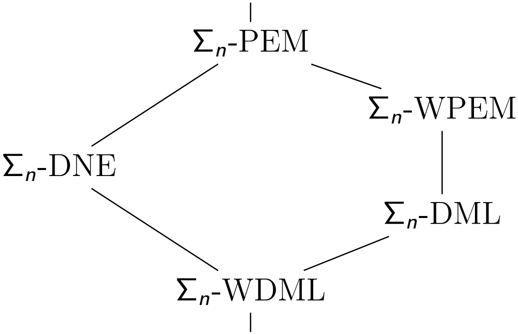
$$\Sigma_1\text{-DML} : \neg(\exists x A_0(x) \wedge \exists y B_0(y)) \to \neg\exists x A_0(x) \vee \neg\exists y B_0(x);$$

$$\Sigma_1\text{-DNE} : \neg\neg\exists x A_0(x) \to \exists x A_0(x).$$

$$\Sigma_1\text{-WPEM} : \neg\exists x A_0(x) \vee \neg\neg\exists x A_0(x);$$

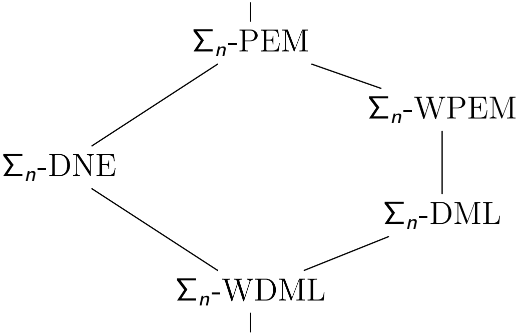
$$\Sigma_1\text{-WDML} : \neg(\neg\exists x A_0(x) \wedge \neg\exists y B_0(y)) \to \neg\neg\exists x A_0(x) \vee \neg\neg\exists y B_0(x).$$

Arithmetical hierarchy over HA + Σ_{n-1}-DNE



Remark. In constructive mathematics, Σ_1-PEM, Σ_1-DML and Σ_1-DNE are known as **LPO**, **LLPO** and Markov's principle respectively.

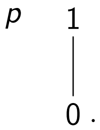
Arithmetical hierarchy over HA + Σ_{n-1}-DNE



By sophisticated use of proof interpretations and models of finite-type arithmetic, one can obtain a lot of saparation results between the Σ_1-fragements of the logical principles over HA.

Separation by Kripke Models (for $n = 1$)

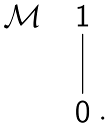
$\neg\neg p \to p$ is not valid in IPC-Kripke model $\mathcal{K}_0 = (K_0, \leq_0, \Vdash_0)$:



On the other hand, $\neg q \vee \neg\neg q$ is valid in any Kripke model with the frame K_0.

A Proof of HA + $\mathrm{WLEM} \nvdash \Sigma^0_1\text{-}\mathrm{DNE}$

- Let $\exists x A_0(x)$ be $\exists x\, \mathsf{Proof}_{\mathsf{PA}}(\mathsf{x}, \ulcorner 0 = 1 \urcorner)$.
- Consider the following Kripke model $\mathcal{K}_0^{\mathsf{HA}}$ of HA:



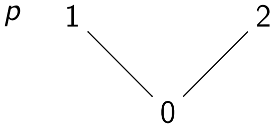
where the domain and evaluation for 0 are given by the standard model ω of PA and those for 1 are given by a non-standard model $\mathcal{M}$ of PA $+ \exists x A_0(x)$.

- Then $\mathcal{K}_0^{\mathsf{HA}} \nVdash \neg\neg\exists x A_0(x) \to \exists x A_0(x)$.
- On the other hand, a schema $\neg\varphi \vee \neg\neg\varphi$ is valid in $\mathcal{K}_0^{\mathsf{HA}}$.
- By the soundness of Kripke semantics for intuitionistic predicate logic, we have HA $+ \mathrm{WPEM} \nvdash \Sigma^0_1\text{-}\mathrm{DNE}$.

□

Separation by a Kripke Model refuting WPEM

$\neg p \vee \neg\neg p$ is not valid in IPC-Kripke model $\mathcal{K}_1 = (K_1, \leq_1, \Vdash_1)$:

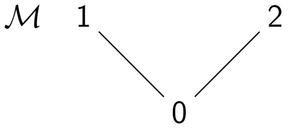


.

On the other hand, $\neg(q \wedge r) \to \neg q \vee \neg r$ and $\neg\neg s \to s$ are valid in any Kripke model with the frame K_1 such that the evaluation for 2 is the same as that for 0.

$\mathsf{HA} + \Sigma^0_1\text{-DML} + \Sigma^0_1\text{-DNE} \nvdash \Sigma^0_1\text{-WPEM}$

- Let $\exists x A_0(x)$ be $\exists x \, \mathsf{Proof}_{\mathsf{PA}}(\mathsf{x}, \ulcorner 0 = 1 \urcorner)$.
- Consider the following Kripke model $\mathcal{K}^{\mathsf{HA}}_1$ of HA:

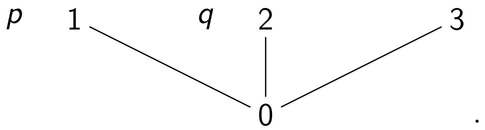


where the domain and evaluation for 0 and 2 are given by the standard model ω of PA and those for 1 are given by a non-standard model $\mathcal{M}$ of $\mathsf{PA} + \exists x A_0(x)$.

- Then $\mathcal{K}^{\mathsf{HA}}_1 \not\Vdash \neg \exists x A_0(x) \vee \neg\neg \exists x A_0(x)$.
- On the other hand, the schemata $\Sigma^0_1\text{-DML}$ and $\Sigma^0_1\text{-DNE}$ (for Σ^0_1-formulae) are valid in $\mathcal{K}^{\mathsf{HA}}_1$.
- By the soundness of Kripke semantics for intuitionistic predicate logic, we have $\mathsf{HA} + \Sigma^0_1\text{-DML} + \Sigma^0_1\text{-DNE} \nvdash \Sigma^0_1\text{-WPEM}$. □

Separation by a Kripke Model refuting DML

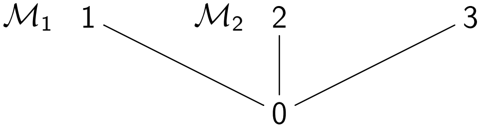
$\neg(p \vee q) \to \neg p \vee \neg q$ is not valid in IPC-Kripke model $\mathcal{K}_2 = (K_2, \leq_2, \Vdash_2)$:



On the other hand, $\neg\neg r \to r$ is valid in any Kripke model with the frame K_2 such that the evaluation for 3 is the same as that for 0.

HA + Σ^0_1-DNE $\nvdash$ Σ^0_1-DML

- Let $\exists x A_0(x)$ and $\exists x B_0(x)$ be Σ_1-formulae which are independent and consistent with PA.
- Consider the following Kripke model $\mathcal{K}_2^{\mathsf{HA}}$ of HA:



  where the domain and evaluation for 0 and 3 are given by ω and those for 1 and 2 are given by non-standard models $\mathcal{M}_1$ and $\mathcal{M}_2$ s.t. $\mathcal{M}_1 \models_c \mathsf{PA} + \exists x A_0(x) + \neg\exists x B_0(x)$ and $\mathcal{M}_2 \models_c \mathsf{PA} + \exists x B_0(x) + \neg\exists x A_0(x)$ respectively.
- $\mathcal{K}_2^{\mathsf{HA}} \nVdash \neg(\exists x A_0(x) \wedge \exists x B_0(x)) \to \neg\exists x A_0(x) \vee \neg\exists x B_0(x)$.
- Σ^0_1-DNE is valid in $\mathcal{K}_2^{\mathsf{HA}}$.
- By the soundness of Kripke semantics for intuitionistic predicate logic, we have HA + Σ^0_1-DNE $\nvdash$ Σ^0_1-DML. □

Last Slide of my Talk at ILLC on 28 June 2019

- An ongoing joint work with Hajime Ishihara, Takako Nemoto, Nobu-Yuki Suzuki and Keita Yokoyama:

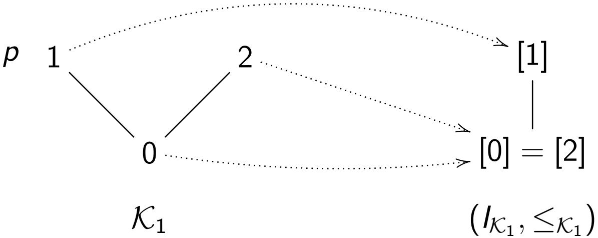
  We are trying to construct a general machinery to apply the simple Kripke models which separate the logical principles in propositional logic for the separations of the Σ^0_1-restrictions of the logical principles in arithmetic.

Meta-theorem. (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted tree and the induced extended frame $\mathcal{E}_{\mathcal{K}}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then for all n,

$$\mathrm{HA} + \Sigma_{n-1}\text{-PEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma_n\text{-}\varphi.$$

A crucial idea underlying this meta-theorem is to restrict possible evaluations on the Kripke frame by using the **extended frame** generated by a given Kripke model.



$[k] := \{k' \in K_1 \mid k \in U \leftrightarrow k' \in U$ for any evaluation set U of $\mathcal{K}_1\}$.

Definition (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

An extended frame $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ is a triple of frames $(K, \leq)$ and $(I, \leq_I)$, and a monotone mapping f between them, that is, $k \leq k'$ implies $f(k) \leq_I f(k')$ for each $k, k' \in K$.

- Each IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$ induces an IPC-Kripke model $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, \Vdash_{\mathcal{E},\mathcal{I}})$ by defining

$$k \Vdash_{\mathcal{E},\mathcal{I}} p :\Leftrightarrow f(k) \Vdash p$$

  for each $k \in K$ and propositional variable p.
- A propositional formula φ is valid on $\mathcal{E}$ if $\mathcal{K}_{\mathcal{E},\mathcal{I}} \Vdash_{\mathcal{E},\mathcal{I}} \varphi$ for each IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$, that is, for each valuation $\Vdash$ on $(I, \leq_I)$; we then write $\mathcal{E} \models \varphi$.
- For an extended frame $\mathcal{E}$, define $T(\mathcal{E}) := \{\varphi \mid \mathcal{E} \models \varphi\}$.

Remark

- For a frame $(K, \leq)$, the set

$$L(K, \leq) = \{\varphi \mid (K, \leq) \models \varphi\}$$

  of propositional formulae is an intermediate propositional logic.
- In contrast, for an extended frame $\mathcal{E}$, $T(\mathcal{E})$ is not an intermediate propositional logic in general. In particular, $T(\mathcal{E})$ may not be closed under substitution.

- Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and define a set $\Phi_{\mathcal{K}}$ of upward closed subsets of K by

$$\Phi_{\mathcal{K}} := \{\{k \in K \mid k \Vdash p\} \mid p \in \mathcal{V}\}.$$

- Define binary relations $\preceq_{\mathcal{K}}$ and $\sim_{\mathcal{K}}$ on K by

$$k \preceq_{\mathcal{K}} k' :\Leftrightarrow k \in U \text{ implies } k' \in U \text{ for all } U \in \Phi_{\mathcal{K}},$$
$$k \sim_{\mathcal{K}} k' :\Leftrightarrow k \preceq_{\mathcal{K}} k' \text{ and } k' \preceq_{\mathcal{K}} k.$$

- Then $\preceq_{\mathcal{K}}$ is a preorder and $\sim_{\mathcal{K}}$ is an equivalence relation on K.
- Let $I_{\mathcal{K}} := K/\sim_{\mathcal{K}}$, $[k]_{\mathcal{K}} \leq_{\mathcal{K}} [k']_{\mathcal{K}} :\Leftrightarrow k \preceq_{\mathcal{K}} k'$, $f_{\mathcal{K}}(k) := [k]_{\mathcal{K}}$, where $[k]_{\mathcal{K}}$ (we sometimes suppress the subscript $\mathcal{K}$) is the equivalence class of k w.r.t. $\sim_{\mathcal{K}}$.
- Then $\mathcal{E}_{\mathcal{K}} := ((K, \leq), f_{\mathcal{K}}, (I_{\mathcal{K}}, \leq_{\mathcal{K}}))$ is an extended frame, and we call it the extended frame generated by the IPC-Kripke model $\mathcal{K}$.

Definition

- For a propositional formula $\varphi[p_1, \ldots, p_m]$, Σ_n-φ denotes a schema $\varphi[\chi_1/p_1, \ldots, \chi_m/p_m]$, where $\chi_1, \ldots, \chi_m$ are Σ_n-formulae of HA, and Σ-φ denotes the following schema of HA :

$$\forall x(\psi_1(x) \vee \neg\psi_1(x)) \wedge \ldots \wedge \forall x(\psi_m(x) \vee \neg\psi_m(x))$$
$$\rightarrow \varphi[\exists x \psi_1(x)/p_1, \ldots, \exists x \psi_m(x)/p_m].$$

- For an extended frame $\mathcal{E}$, Σ-$T(\mathcal{E})$ is the schema (of HA) consisting of Σ-φ where $\varphi \in T(\mathcal{E})$.
- For $k \in K$, let $\uparrow k$ denote $\{k' \in K \mid k \leq k'\}$.
- An extended frame $\mathcal{E}$ is locally directed if $f^{-1}(\uparrow i) \cap \uparrow k$ is directed for all $i \in I$ and $k \in K$, that is, for each $i \in I$ and $k \in K$, if $l, l' \in f^{-1}(\uparrow i) \cap \uparrow k$, then there exists $l'' \in f^{-1}(\uparrow i) \cap \uparrow k$ such that $l'' \leq l$ and $l'' \leq l'$.

Meta-theorem. (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted tree and the induced extended frame $\mathcal{E}_{\mathcal{K}}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then for all n,

$$\mathrm{HA} + \Sigma_{n-1}\text{-}\mathrm{PEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma_n\text{-}\varphi,$$

where $L(K, \leq)^*$ is the set of schemata of $\varphi[\psi_1/p_1, \ldots, \psi_m/p_m]$ for propositional formulae $\varphi[p_1, \ldots, p_m] \in L(K, \leq)$.

Meta-theorem. (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted tree and the induced extended frame $\mathcal{E}_{\mathcal{K}}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then for all n,

$$\mathsf{HA} + \Sigma_{n-1}\text{-}\mathrm{PEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma_n\text{-}\varphi,$$

where $L(K, \leq)^*$ is the set of schemata of $\varphi[\psi_1/p_1, \ldots, \psi_m/p_m]$ for propositional formulae $\varphi[p_1, \ldots, p_m] \in L(K, \leq)$.

Corollary. (De Jongh's theorem)

If $\varphi[p_1 \ldots, p_m] \notin \mathsf{IPC}$, then $\mathsf{HA} \nvdash \varphi[\chi_1/p_1, \ldots, \chi_m/p_m]$ for some Σ_1-formulae $\chi_1, \ldots, \chi_m$ of HA.

Meta-theorem. (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted tree and the induced extended frame $\mathcal{E}_{\mathcal{K}}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then for all n,

$$\mathsf{HA} + \Sigma_{n-1}\text{-}\mathrm{PEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma_n\text{-}\varphi,$$

where $L(K, \leq)^*$ is the set of schemata of $\varphi[\psi_1/p_1, \ldots, \psi_m/p_m]$ for propositional formulae $\varphi[p_1, \ldots, p_m] \in L(K, \leq)$.

Corollary. (De Jongh's theorem)

If $\varphi[p_1 \ldots, p_m] \notin \mathsf{IPC}$, then $\mathsf{HA} \nvdash \varphi[\chi_1/p_1, \ldots, \chi_m/p_m]$ for some Σ_1-formulae $\chi_1, \ldots, \chi_m$ of HA.

Observation.

The Σ_n-substitution instances of $\mathrm{PEM}, \mathrm{WPEM}, \mathrm{DML}$, WDML, DNE can be separated uniformly by the technique.

A Case Study: 6 Linearity Axioms

$$\mathrm{LIN}_1 : (\varphi \to \psi) \vee (\psi \to \varphi);$$
$$\mathrm{LIN}_2 : (\varphi \to \neg\psi) \vee (\neg\psi \to \varphi).$$
$$\mathrm{LIN}_3 : (\neg\varphi \to \neg\psi) \vee (\neg\psi \to \neg\varphi);$$
$$\mathrm{LIN}_4 : (\neg\varphi \to \neg\neg\psi) \vee (\neg\neg\psi \to \neg\varphi);$$
$$\mathrm{LIN}_5 : (\neg\neg\varphi \to \neg\neg\psi) \vee (\neg\neg\psi \to \neg\neg\varphi);$$
$$\mathrm{LIN}_6 : (\varphi \to \neg\neg\psi) \vee (\neg\neg\psi \to \varphi).$$

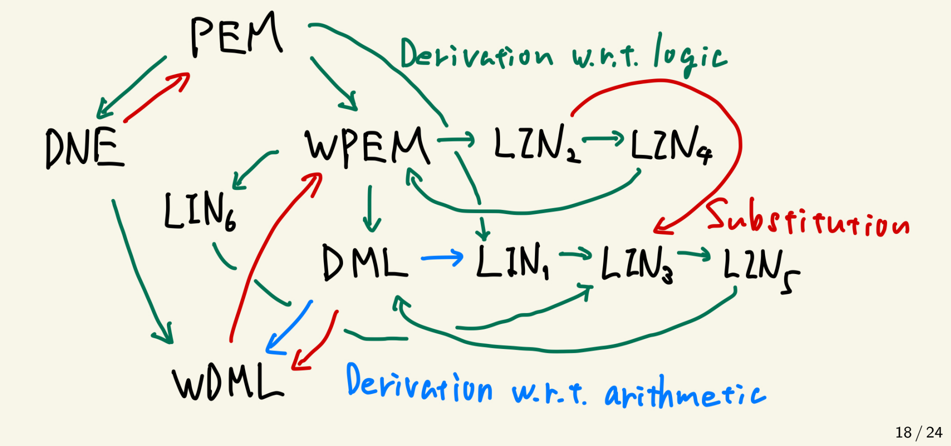
Fact. (Hierarchy of Intermediate Propositional Logics)

$$\mathrm{PEM} = \mathrm{DNE} \supsetneq \mathrm{LIN}_1 \supsetneq \mathrm{WPEM} = \mathrm{DML} = \mathrm{WDML} = \mathrm{LIN}_2 = \mathrm{LIN}_3 = \mathrm{LIN}_4 = \mathrm{LIN}_5 = \mathrm{LIN}_6.$$

Derivations and Substitutions

A set **L** of propositional formulae s.t. IPC $\subseteq$ **L** $\subseteq$ CPC is called **intermediate propositional logic** if the following hold:

1. if $\varphi \to \psi$ and φ are in **L**, then ψ is in **L**;
2. if φ is in **L**, then any substitution instance of φ is in **L**.



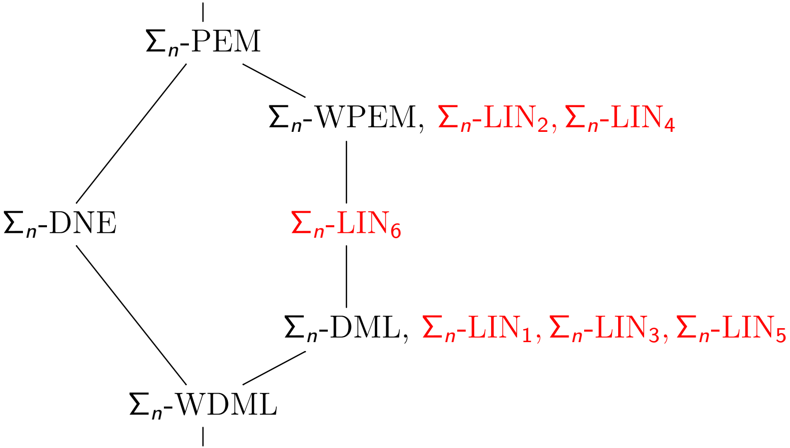
$\mathsf{HA} + \Sigma_n\text{-DML} + \Sigma_{n-1}\text{-DNE} \vdash \Sigma_n\text{-LIN}_1$.

Proof. Fix Σ_n-formulae φ_1 and φ_2. W.l.o.g, assume $n > 0$. We show $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_1)$ within $\mathsf{HA} + \Sigma_n\text{-DML} + \Sigma_{n-1}\text{-DNE}$. Let φ_1 and φ_2 be $\exists x \varphi_1'(x)$ and $\exists x \varphi_2'(x)$ where $\varphi_1'(x)$ and $\varphi_2'(x)$ are Π_{n-1}-formulae respectively. Consider the following formulae:

$$\begin{aligned} \psi_1(x) &\equiv \quad \varphi_1'(x) \wedge \forall y \leq x \neg \varphi_2'(y); \\ \psi_2(x) &\equiv \quad \varphi_2'(x) \wedge \forall y \leq x \neg \varphi_1'(y). \end{aligned}$$

Then we have $\mathsf{HA} \vdash \neg(\exists x \psi_1(x) \wedge \exists x \psi_2(x))$ trivially. Since $\neg \varphi_2'(y)$ and $\neg \varphi_1'(y)$ are equivalent to some Σ_{n-1}-formulae respectively in the presence of Σ_{n-1}-DNE, we have that$\forall y \leq x \neg \varphi_2'(y)$ and $\forall y \leq x \neg \varphi_1'(y)$ are equivalent to some Σ_{n-1}-formulae respectively. Therefore we have that $\exists x \psi_1(x)$ and $\exists x \psi_2(x)$ are equivalent to some Σ_n-formulae respectively in our theory. Applying Σ_n-DML, we have $\neg \exists x \psi_1(x) \vee \neg \exists x \psi_2(x)$. In the former case, if $\varphi_1'(x)$, then we have $\neg \forall y \leq x \neg \varphi_2'(y)$, equivalently, $\neg\neg \exists y \leq x \varphi_2'(y)$. Then we have $\exists y \leq x \varphi_2'(y)$ by using Σ_{n-1}-DML and Σ_{n-2}-DNE. Thus we have shown $\exists x \varphi_1'(x) \to \exists x \varphi_2'(x)$. In the latter case, we have $\exists x \varphi_2'(x) \to \exists x \varphi_1'(x)$ similarly. □

Arithmetical hierarchy over HA + Σ_{n-1}-DNE



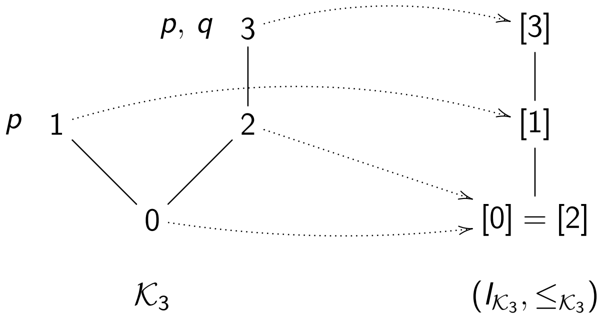
$$\Sigma_1\text{-LIN}_6 : (\exists x A_0(x) \to \neg\neg\exists y B_0(y)) \lor (\neg\neg\exists y B_0(y) \to \exists x A_0(x)).$$

Theorem.

$\mathrm{HA} + \Sigma_n\text{-DML} + \Sigma_{n-1}\text{-PEM} \nvdash \Sigma_n\text{-LIN}_6$.

Proof Sketch.

Apply the above-mentioned meta-theorem to the IPC-Kripke model $\mathcal{K}_3 = (K_3, \leq_3, \Vdash_3)$ given in the following figure:



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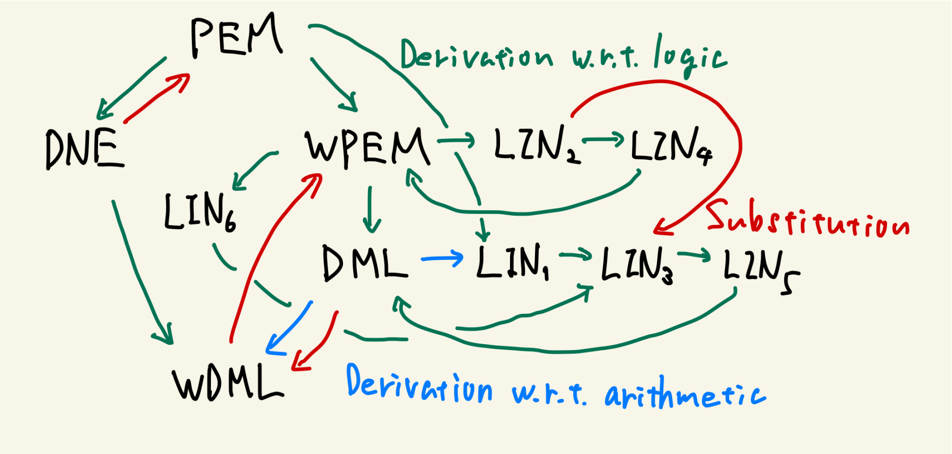
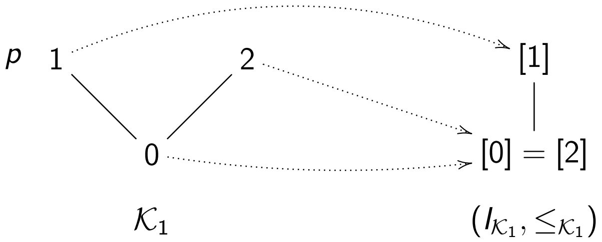
Hierarchy of Our Axioms w.r.t Logic & Arithmetic



An Example which is not an Intermediate Logic

The theory $T(\mathcal{E}_{\mathcal{K}_1})$ generated by the previous IPC-Kripke model $\mathcal{K}_1$ is not closed under substitution:



- It is straightforward to see $\mathrm{LIN}_1 \in T(\mathcal{E}_{\mathcal{K}_1})$.
- $\mathrm{LIN}_2 \notin T(\mathcal{E}_{\mathcal{K}_1})$:
  Consider a IPC-Kripke model $\mathcal{I} := (I_{\mathcal{K}_1}, \leq_{\mathcal{K}_1}, \Vdash)$ with $[0] \nVdash q, r$ but $[1] \Vdash q, r$. Then we have

$$0 \nVdash_{\mathcal{E}_{\mathcal{K}_1}, \mathcal{I}} (q \to \neg r) \vee (\neg r \to q).$$

References

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