



# From Bi-nested Sequents to Shallow Sequents

proof theory of the intuitionistic modal logic FIK

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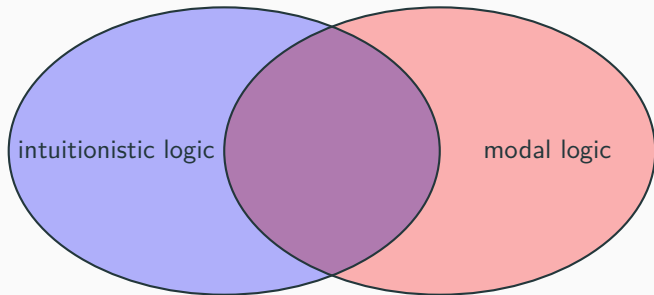
**Han GAO** [ghlogic123.github.io](https://github.com/ghlogic123)

(Institute of Computer Science, Czech Academy of Sciences)

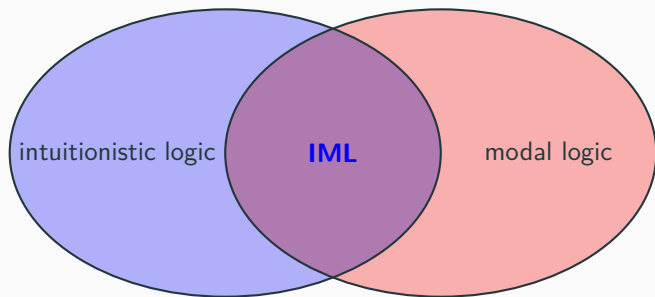
Based on joint work with Philippe Balbiani, Çiğdem Gencer and Nicola Olivetti

LLAMA seminar @ ILLC, Amsterdam - 20<sup>th</sup> May, 2026

# What is intuitionistic modal logic



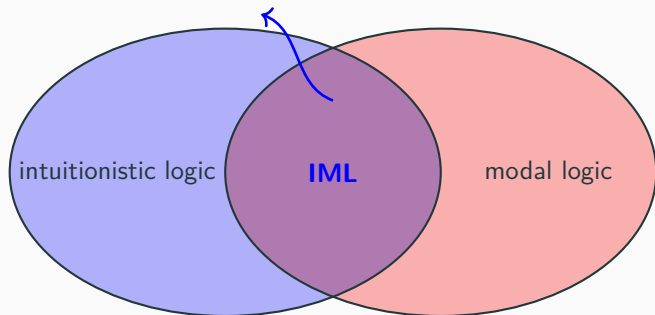
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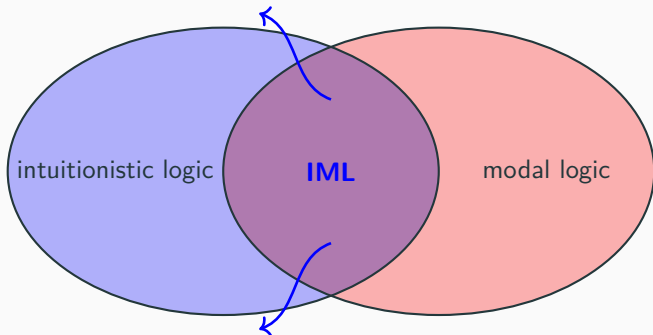
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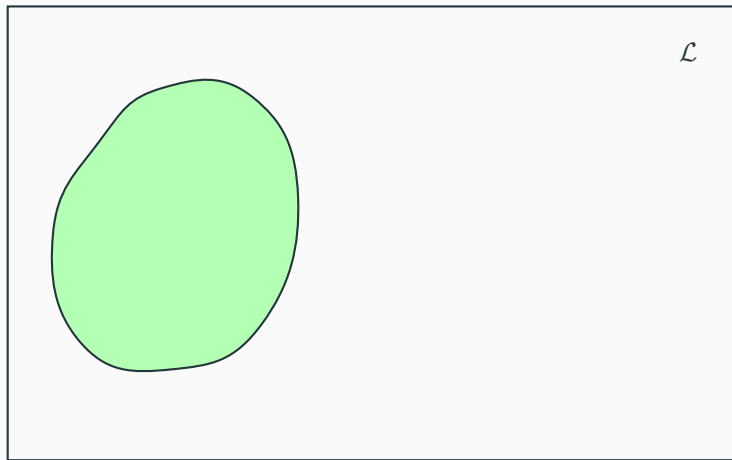


Constructive modal logic (CML), motivated by applications in CS,

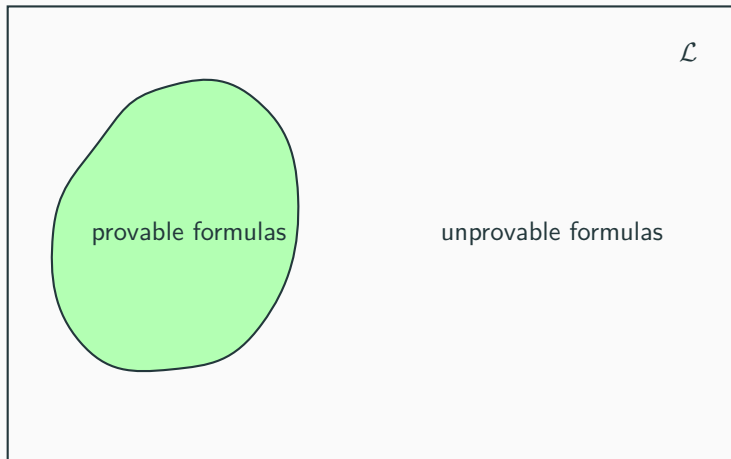
- type theory, verification and knowledge representation, contextual reasoning, see e.g. [de Paiva, 2003, Bellin et al., 2001, Mendler and de Paiva, 2005]

- systems: **CK** and **WK** [Wijesekera, 1990, Bellin et al., 2001]

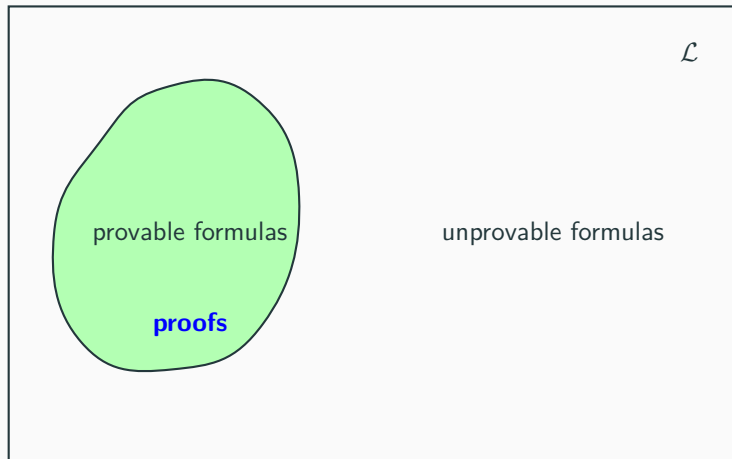
## Provability v.s. Unprovability



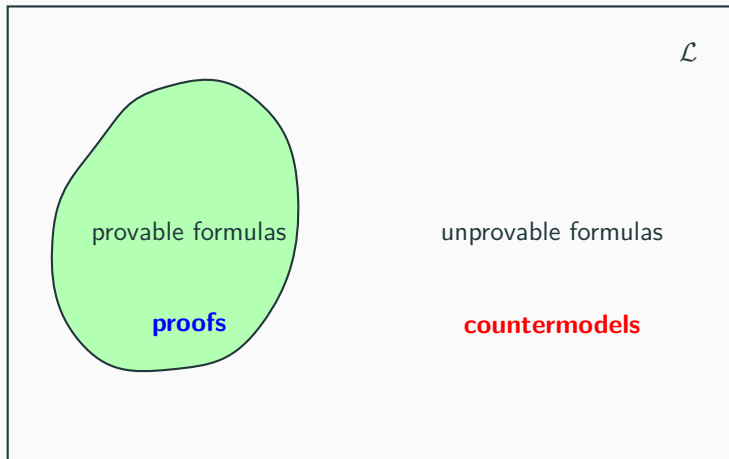
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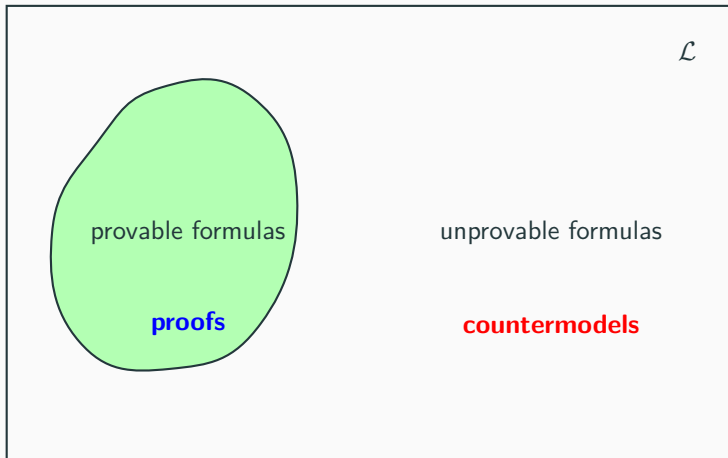
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we view **countermodels** as constructive witnesses of unprovability

# Outline

1. Background
2. A natural intuitionistic modal logic
3. A proof-theoretic framework: bi-nested sequent calculus
4. Shallow sequent calculus and complexity
5. Further results and perspectives

# Background

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## Intuitionistic logic

- Language:

$$p \in \text{At} \mid \perp \mid (A \vee A) \mid (A \wedge A) \mid (A \supset A)$$

- Kripke model:

$$\mathcal{M} = (W, \leq, V) \quad x, x' \in W$$

- $W$  non-empty set
  - $\leq$  pre-order
  - $V : W \rightarrow \mathcal{P}(\text{At})$  with monotonicity
- Semantics:  
 $x \Vdash A \supset B$  iff  
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# Kripke semantics for IPL and ML

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- propositional connectives are interpreted as in **IPL**
- interpretations for modalities vary in different systems
- **HP** must be preserved

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## Modalities in CK/WK [Bellin et al., 2001, Wijesekera, 1990]

**global**  $\square$   $\mathcal{M}, x \Vdash \square A$  if and only if  $\forall y. x \leq y \forall z. (Ryz \Rightarrow z \Vdash A)$

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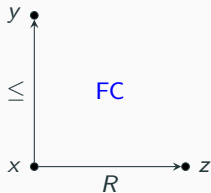
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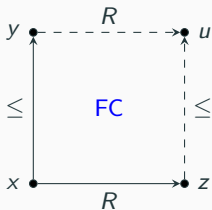
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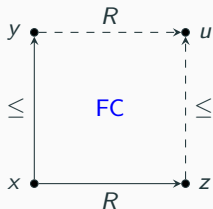
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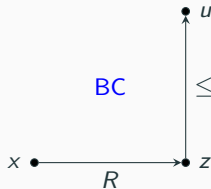
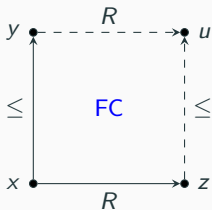
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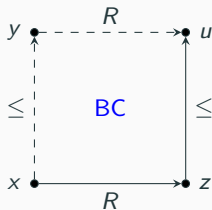
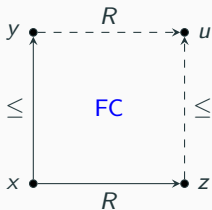
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## Hilbert axiom systems

The axiom system  $\mathbb{H}_{\mathbf{IK}}$  for  $\mathbf{IK}$  contains:

$$(k1/K_{\Box}) \quad \Box(A \supset B) \supset (\Box A \supset \Box B)$$

$$(k2/K_{\Diamond}) \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$$

$$(k3/DP) \quad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$$

$$(k4/FS) \quad (\Diamond A \supset \Box B) \supset \Box(A \supset B)$$

$$(k5/N) \quad \Diamond \perp \supset \perp$$

$$(MP) \quad \frac{A \quad A \supset B}{B}$$

$$(Nec) \quad \frac{A}{\Box A}$$

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The axiom system  $\mathbb{H}_{CK}$  for **CK** contains:

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- $\mathbf{WK}^+$  and diamond-free fragments of some IMLs: [Das and Marin, 2023]
- frame conditions for logics from **CK** to **IK**: [de Groot et al., 2025]

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## Theorem (Soundness and completeness)

Let  $\mathbf{X} \in \{\mathbf{CK}, \mathbf{WK}, \mathbf{WK}^+, \mathbf{IK}\}$ .  $\mathbb{H}_{\mathbf{X}}$  is sound and complete w.r.t. the bi-relational semantics.

# Existing systems



Informal discussions on logics between **CK** and **IK**:

## The Proof Theory Blog

$$\alpha < \beta \Rightarrow \varphi\alpha(\varphi\beta\gamma) = \varphi\beta\gamma$$

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### Diamond-free parts of intuitionistic modal logics

Posted on October 3, 2025 by Anupam Das, Ian Shillito and Jim de Groot

link: <https://prooftheory.blog/2025/10/03/diamond-free-parts-of-intuitionistic-modal-logics/>

# **A natural intuitionistic modal logic**

---

# What makes a good IML

In [Simpson, 1994], Simpson proposed some criteria for a good IML:

1. it is a conservative extension of IPL
2. it satisfies the disjunction property
3. modalities are normal and independent
4. by adding  $A \vee \neg A$ , it yields a classical modal logic
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## Remarks on Simpson's criteria

- **WK/CK**: not 'good' as it does not satisfy e.g. (2)-(4)
- **IK**: 'good' as it fits all the criteria (1)-(5)
- (5) is controversial in our opinion
- something else?

## Motivation: to find more 'good' IMLs

**Question: can we find other 'good' IMLs with naturally defined semantics and similar to IK?**

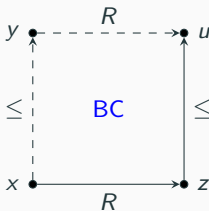
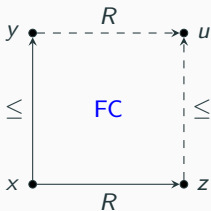
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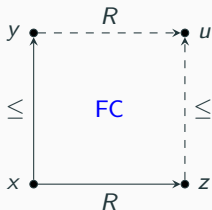


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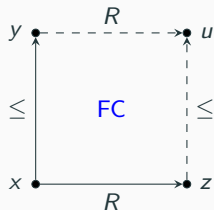
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- **Question: what is the logic of the class (FC)-frames?**

## A natural intuitionistic modal logic: **FIK**

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## Axiom system $\mathbb{H}_{\text{FIK}}$

- $\mathbb{H}_{\text{WK}}$
- (DP)  $\diamond(A \vee B) \supset \diamond A \vee \diamond B$
- (**wCD**)  $\Box(A \vee B) \supset ((\diamond A \supset \Box B) \supset \Box B)$

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- (**wCD**)  $\Box(A \vee B) \supset ((\diamond A \supset \Box B) \supset \Box B)$

## Theorem (Soundness and completeness)

$\mathbb{H}_{\text{FIK}}$  is sound and complete w.r.t. the bi-relational semantics.

# A natural intuitionistic modal logic: FIK

Let **FIK** be the logic of the class of (FC)-frames.

- In **FIK**,  $\diamond$  is local, i.e.

$$x \Vdash \diamond A \text{ if and only if } \exists y.(Rxy \ \& \ y \Vdash A)$$

- it seems to be the minimal logic defined by bi-relational models with global  $\Box$  and local  $\diamond$  preserving **HP**

## Axiom system $\mathbb{H}_{\text{FIK}}$

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▷ **FIK** satisfies all the criteria proposed by Simpson except the last one

# Hierarchy of IMLs, plus FIK



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# **A proof-theoretic framework: bi-nested sequent calculus**

---

## Existing calculi for IMLs

- for **WK** and **CK**:

simple Gentzen calculi [Wijesekera, 1990, Bellin et al., 2001]

G4-calculi with countermodel w.r.t. neighbourhood semantics [Dalmonte et al., 2021]

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Single-conclusion, nested	[Straßburger, 2013, Galmiche and Salhi, 2010]
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## A 'suitable' proof-theoretic framework in our view

- terminating proof-search is supported
- a countermodel can be extracted directly from a failed proof
- different IMLs can be treated in a modular way

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**bi-nested sequent calculus!**

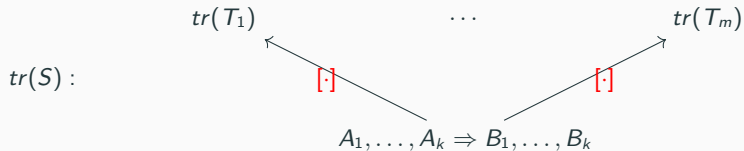
## From sequent to bi-nested sequent

- ▶ **simple sequent**:  $\Gamma \Rightarrow \Delta$ , ordered pair of multisets of formulas



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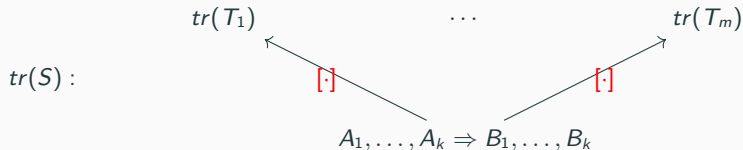
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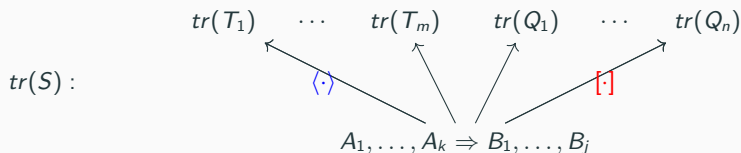
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► **bi-nested sequent:**

$S = A_1, \dots, A_k \Rightarrow B_1, \dots, B_j, \langle T_1 \rangle, \dots, \langle T_m \rangle, [Q_1], \dots, [Q_n]$



# Bi-nested sequent calculus

- two types of nesting are used, roughly

$[\cdot]$  simulating  $R$ , see [Brünnler, 2009, Poggiolesi, 2009]  
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e.g.

$$\frac{G\{\Gamma \Rightarrow \Delta, \langle A \Rightarrow B \rangle\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}} (\supset_R) \quad \frac{G\{\Gamma \Rightarrow \Delta, \diamond A, [\Sigma \Rightarrow \Pi, A]\}}{G\{\Gamma \Rightarrow \Delta, \diamond A, [\Sigma \Rightarrow \Pi]\}} (\diamond_R)$$

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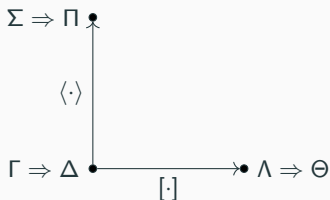
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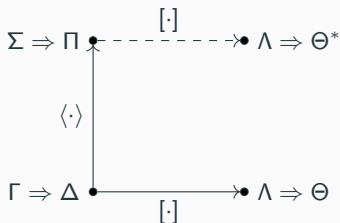
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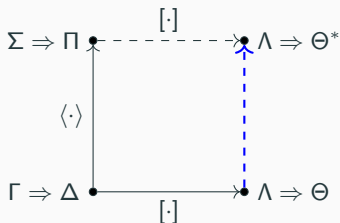
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# The calculus $C_{FIK}$

$$\frac{}{G\{\Gamma, \perp \Rightarrow \Delta\}} (\perp_L)$$

$$\frac{G\{A, B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}} (\wedge_L)$$

$$\frac{G\{\Gamma, A \Rightarrow \Delta\} \quad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \vee B \Rightarrow \Delta\}} (\vee_L)$$

$$\frac{G\{\Gamma, A \supset B \Rightarrow A, \Delta\} \quad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \supset B \Rightarrow \Delta\}} (\supset_L)$$

$$\frac{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]\}}{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}} (\Box_L)$$

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where the left leaf is obtained from

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The formula  $(\neg \Box \perp \supset \Box \perp) \supset \Box \perp$  is provable in  $C_{\text{FIK}}$ .

Let  $F = \neg \Box \perp \supset \Box \perp$  and we prove  $F \Rightarrow \Box \perp$ .

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# Semantic interpretation

## Definition (Interpretation)

Let  $\mathcal{M} = (W, \leq, R, V)$  be a bi-relational model and  $w \in W$ . The satisfiability relation  $\Vdash$  is extended to sequents as follows:

$\mathcal{M}, w \not\Vdash \emptyset$

$\mathcal{M}, w \Vdash [T]$  if for every  $v$  with  $Rwv$ ,  $\mathcal{M}, v \Vdash T$

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$\mathcal{M}, w \Vdash \Gamma \Rightarrow \Delta$  if either  $\mathcal{M}, w \not\Vdash A$  for some  $A \in \Gamma$  or  $\mathcal{M}, w \Vdash \mathcal{O}$  for some  $\mathcal{O} \in \Delta$ , where  $\mathcal{O} \in \mathcal{L}$  or is a block.

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## Theorem (Soundness)

*If a formula  $A$  is provable in  $\mathbf{C}_{\text{FIK}}$ , then it is valid.*

## Towards termination: saturation mechanism

We introduce a set-based variant of the calculus where all rules are cumulative, e.g.

$$\frac{G\{A, B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}} (\wedge_L) \quad \hookrightarrow \quad \frac{G\{A, B, A \wedge B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}} (\wedge_L)$$

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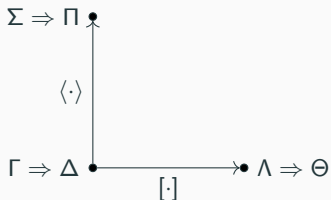
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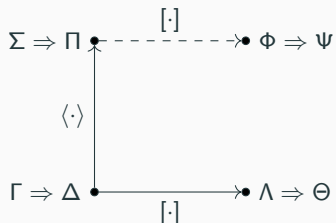
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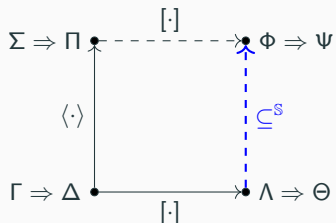
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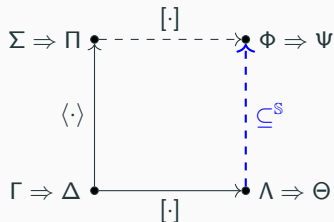
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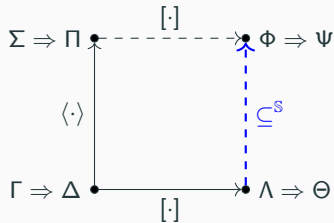
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► it will be taken as the pre-order  $\leq$  in the countermodel construction

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Let  $\mathcal{M}_S$  be defined as above. We claim

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## Semantic completeness of $\mathbf{C}_{\text{FIK}}$

For any formula  $A \in \mathcal{L}$ , if  $\Vdash A$ , then  $\Rightarrow A$  is provable in  $\mathbf{C}_{\text{FIK}}$ .

## Step 1: obtain a saturated leaf

▶  $F = (\diamond p \supset \Box q) \supset \Box(p \supset q)$  is valid in **IK** but not in **FIK**.





## Step 2: countermodel construction

$$S_0 = \langle \Diamond p \supset \Box q \Rightarrow \Diamond p, \Box(p \supset q), \langle \Diamond p \supset \Box q \Rightarrow \Diamond p, [\Rightarrow p \supset q, \langle p \Rightarrow q \rangle, p] \rangle \rangle$$

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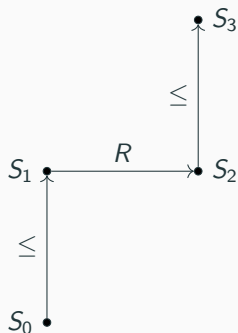
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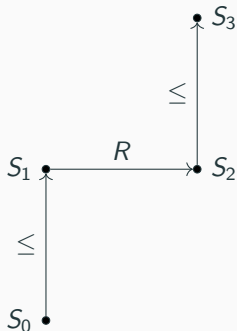
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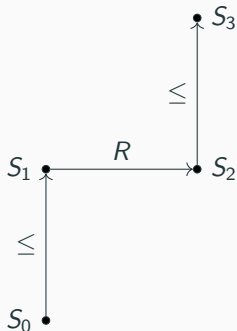
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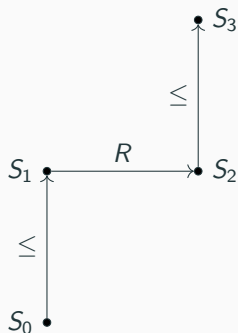
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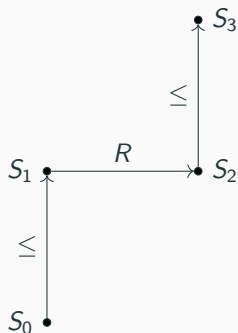
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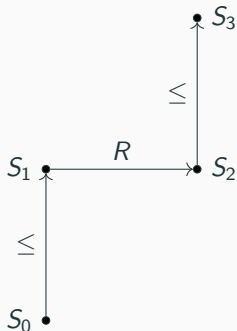
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# Shallow sequent calculus and complexity

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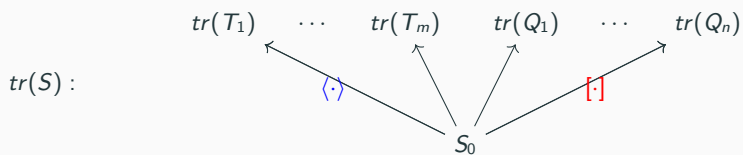
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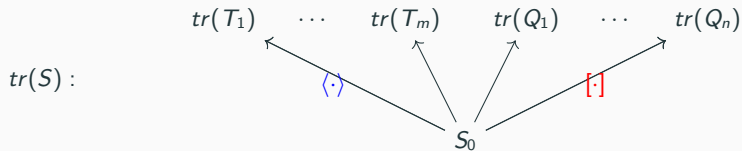
# From bi-nested sequents to shallow sequents

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- **shallow sequent:**

$$\Phi, [\Psi_1], \dots, [\Psi_l]$$

where exactly one of  $\Phi, \Psi_1, \dots, \Psi_l$  is a full sequent and others are left-sided sequents

# A shallow calculus for FIK

$$\begin{array}{c}
 \frac{}{\Omega, \perp^\bullet} (\perp_1^\bullet) \quad \frac{}{\Omega, [\Phi, \perp^\bullet]} (\perp_2^\bullet) \quad \frac{}{\Omega, \rho^\bullet, \rho^\circ} (\text{id}_1) \quad \frac{}{\Omega, [\Phi, \rho^\bullet, \rho^\circ]} (\text{id}_2) \\
 \\
 \frac{\Omega, A^\bullet, B^\bullet}{\Omega, A \wedge B^\bullet} (\wedge_1^\bullet) \quad \frac{\Omega, [\Phi, A^\bullet, B^\bullet]}{\Omega, [\Phi, A \wedge B^\bullet]} (\wedge_2^\bullet) \quad \frac{\Omega, A_i^\circ}{\Omega, A_1 \vee A_2^\circ} (\vee_1^\circ) \quad \frac{\Omega, [\Phi, A_i^\circ]}{\Omega, [\Phi, A_1 \vee A_2^\circ]} (\vee_2^\circ) \\
 \\
 \frac{\Omega, A^\circ}{\Omega, A \wedge B^\circ} (\wedge_1^\circ) \quad \frac{\Omega, B^\circ}{\Omega, [\Phi, B^\circ]} (\wedge_2^\circ) \\
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 \\
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 \frac{\Omega, \Box A^\bullet, [\Phi, A^\bullet]}{\Omega, \Box A^\bullet, [\Phi]} (\Box^\bullet) \quad \frac{\Omega, [A^\circ]}{\Omega, \Box A^\circ} (\Box_1^\circ) \quad \frac{\Phi, [A^\circ]}{\Omega, [\Phi, \Box A^\circ]} (\Box_2^\circ) \\
 \\
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 \end{array}$$

Figure 1: The calculus  $\text{SC}_{\text{FIK}}$

# Example of derivations

## Example

Axiom (**wCD**) :  $\Box(p \vee q) \supset ((\Diamond p \supset \Box q) \supset \Box q)$  is provable in  $SC_{\text{FIK}}$ .

$$\begin{array}{c}
 \frac{\frac{\Box(p \vee q)^*, \Box q^*, [q^*, p^*, q^\circ]}{\Box(p \vee q)^*, \Box q^*, [p^*, q^\circ]} \Box^\bullet \quad \frac{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, [p^*, p^\circ]}{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, [p^*]} \Diamond^\circ}{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, [p^*, q^\circ]} \supset_i^\circ \quad \frac{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, [q^*, q^\circ]}{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, [p \vee q^*, q^\circ]} \vee_2^\bullet}{\frac{\frac{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, [q^\circ]}{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, \Box q^\circ} \Box^\bullet \quad \frac{\Box(p \vee q)^*, \Diamond p \supset \Box q^*, \Box q^\circ}{\Box(p \vee q)^*, (\Diamond p \supset \Box q) \supset \Box q^\circ} \supset_i^\circ}{\Box(p \vee q) \supset ((\Diamond p \supset \Box q) \supset \Box q)^\circ} \supset_i^\circ}
 \end{array}$$

## Definition (Semantic interpretation)

Let  $\mathcal{M} = (W, \leq, R, V)$  be a bi-relational model and  $x, y \in W$ . For a sequent  $\Omega$ , we define  $\mathcal{M}, x \Vdash \Omega$  according to the location of its output formula as follows

- a. if  $\Omega = \Phi, \overline{[\Psi_i]_{i \in I}}, F^\circ$  where  $\Phi$  is a multi-set of  $\bullet$ -formulas,

$$x \Vdash \Omega \text{ iff } x \Vdash \bigwedge \Phi \wedge \bigwedge \{\diamond \wedge \Psi_i\}_{i \in I} \supset F$$

- b. if  $\Omega = \Phi, \overline{[\Psi_i]_{i \in I}}, [\exists, F^\circ]$  where  $\Phi$  is a multi-set of  $\bullet$ -formulas,

$$x \Vdash \Omega \text{ iff } \forall x' \geq x \text{ if } x' \Vdash \bigwedge \Phi \wedge \bigwedge \{\diamond \wedge \Psi_i\}_{i \in I} \text{ then } \forall y (Rx'y \ \& \ y \Vdash \bigwedge \exists \text{ implies } y \Vdash F)$$

$\Omega$  is called valid if  $\mathcal{M}, x \Vdash \Omega$  holds for any  $\mathcal{M}$  and any  $x$ .

# Semantic interpretation and soundness

## Definition (Semantic interpretation)

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b. if  $\Omega = \Phi, \overline{[\Psi_i]_{i \in I}}, [\Xi, F^\circ]$  where  $\Phi$  is a multi-set of  $\bullet$ -formulas,

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$\Omega$  is called valid if  $\mathcal{M}, x \Vdash \Omega$  holds for any  $\mathcal{M}$  and any  $x$ .

## Theorem (Soundness)

*For any sequent  $\Omega$ , if it is derivable in  $SC_{\text{FIK}}$ , then it is valid in **FIK**.*

## (hp-)Invertibility and admissibility

invertible rules	$(\wedge^\bullet)(\wedge^\circ)(\vee^\bullet)(\Box^\bullet)(\supset_1^\circ)(\Box_1^\circ)(\Diamond_1^\bullet)$
non-invertible rules	$(\vee^\circ)(\supset_2^\circ)(\Box_2^\circ)(\Diamond^\circ)(\Diamond_2^\bullet)$
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The following structural rules are height-preserving admissible in  $SC_{\text{FIK}}$ :

$$\begin{array}{cccc}
 \frac{\Omega}{A^\bullet, \Omega} (w_1^\bullet) & \frac{\Omega, [\Phi]}{\Omega, [A^\bullet, \Phi]} (w_2^\bullet) & \frac{\Omega}{\Omega, [\Phi]} (w^{[\cdot]^+}) & \\
 \frac{A^\bullet, A^\bullet, \Omega}{A^\bullet, \Omega} (c_1^\bullet) & \frac{\Omega, [A^\bullet, A^\bullet, \Phi]}{\Omega, [A^\bullet, \Phi]} (c_2^\bullet) & \frac{\Omega, [\Phi], [\Phi^*]}{\Omega, [\Phi]} (c^{[\cdot]^-}) & \frac{\Phi}{\Omega, [\Phi]} ([\cdot])
 \end{array}$$

## Definition (Cut-rules)

$$\frac{\Omega^*, A^\circ \quad \Omega, A^\bullet}{\Omega} \text{ (cut)}$$

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## Theorem (Cut-admissibility)

*Both (cut) and (cut<sup>[1]</sup>) are admissible in  $SC_{\text{FIK}}$ , meaning that if a sequent  $S$  is derivable in  $SC_{\text{FIK}} + \text{cut} + \text{cut}^{[1]}$  it is also derivable in  $SC_{\text{FIK}}$ .*

## Towards complexity I: 2-EXPSpace

### Definition ((quasi-)set-sequents)

Let  $\Omega$  be a sequent of the form  $\Phi, [\Psi_1], \dots, [\Psi_n]$ .  $\Omega$  is called a *quasi-set-sequent* if each  $\Phi, \Psi_i$  is a set (rather than a multiset) of formulas;  $\Omega$  is further called a *set-sequent* if it is a quasi-set-sequent and for  $i \neq j$ ,  $\Psi_i \neq \Psi_j$ .

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## Proposition

Let  $A$  be a formula and  $|A| = \mathcal{O}(n)$ . Then the size of any set-based sequent that may occur in any possible derivation of  $A$  is  $2^{\mathcal{O}(n)}$ , hence the set of all possible sequents that can occur in any possible derivation has cardinality  $2^{2^{\mathcal{O}(n)}}$ .

Basic proof-search:

1. do not apply rules to an axiom,
2. remove duplicated formulas or blocks in order to keep the set-based structure,
3. stop proof-search if the same sequent already appears in the branch.

## Towards complexity II: EXPSPACE

### Definition (tight sequent and clean derivation)

A *tight* sequent is a set-sequent satisfying the additional condition: if there is a block containing the output formula, then the input part of that block is either empty or is not identical to any other block. A derivation (resp. proof) containing only tight sequents is called *clean* derivation (resp. proof).

### Example

$p \vee q^\bullet, [p^\bullet, q^\bullet], [q^\bullet], p^\circ$  is a tight sequent while neither (1)

$p \vee q^\bullet, [p^\bullet, q^\bullet], [q^\bullet, p^\bullet, p^\circ]$  nor (2)  $p \vee q^\bullet, [p^\bullet, q^\bullet], [p^\bullet, q^\bullet], p^\circ$  is.

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### Lemma

*Backward proof search in  $SC_{\text{FIK}}^\times$  for a formula  $A$  terminates with a finite clean derivation/proof where each branch is of exponential length in terms of  $|A|$ .*

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**Open problem: complexity of the decision problem of IK**

## **Further results and perspectives**

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## Bi-nested sequent calculi have been applied to ...

- **LIK** (a variant of IK with both local  $\square$  and  $\diamond$ ), [BGGO, 2024]
- **IK**, [G., Girlando, Olivetti, 2026]
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► All of these calculi support terminating proof search and countermodel construction with respect to bi-relational semantics

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Recall that **IK** is the logic of  $(FC)+(BC)$ , we can provide a rule for  $(BC)$ , which is

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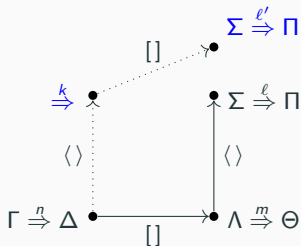
$$\frac{G\{\Gamma \Rightarrow \Delta, [\Lambda \Rightarrow \Theta, \langle \Sigma \Rightarrow \Pi \rangle], \langle \Rightarrow [\Sigma \Rightarrow \Pi] \rangle\}}{G\{\Gamma \Rightarrow \Delta, [\Lambda \Rightarrow \Theta, \langle \Sigma \Rightarrow \Pi \rangle]\}} \text{ (inter}_{bc}\text{)}$$



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- we employ an annotation system to track applications of  $(\text{inter}_{bc})$  in  $\mathbf{C}_{IK}$
- by refining the **FIK** technique to **IK**, we have provided a complete bi-nested calculus for **IK**
- also simulate several existing calculi for **IK**

## Example of derivations

### Example

Axiom (FS):  $(\diamond p \supset \Box q) \supset \Box(p \supset q)$  is provable in  $\mathbf{C}_{IK}$ .

Let  $G_1\{ \} = \diamond p \supset \Box q \Rightarrow \langle \{ \} \rangle$  and  $G_2\{ \} = \diamond p \supset \Box q \Rightarrow [\Rightarrow \langle p \Rightarrow q \rangle], \langle \{ \} \rangle$ .

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$$\begin{array}{c}
 \frac{G_1\{G_2\{\diamond p \supset \Box q \Rightarrow \diamond p, [p \Rightarrow q, p]\}\}}{G_1\{G_2\{\diamond p \supset \Box q \Rightarrow \diamond p, [p \Rightarrow q]\}\}} \text{ (id)} \quad \frac{G_1\{G_2\{\Box q \Rightarrow [q, p \Rightarrow q]\}\}}{G_1\{G_2\{\Box q \Rightarrow [p \Rightarrow q]\}\}} \text{ (id)} \\
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 \frac{\quad}{G_1\{\diamond p \supset \Box q \Rightarrow [\Rightarrow \langle p \Rightarrow q \rangle], \langle \Rightarrow [p \Rightarrow q] \rangle\}} \text{ (trans)} \\
 \frac{\quad}{G_1\{\diamond p \supset \Box q \Rightarrow [\Rightarrow \langle p \Rightarrow q \rangle]\}} \text{ (inter}_{bc}) \\
 \frac{\quad}{\diamond p \supset \Box q \Rightarrow \langle \diamond p \supset \Box q \Rightarrow [\Rightarrow p \supset q] \rangle} \text{ } \quad \text{ } \\
 \frac{\quad}{\diamond p \supset \Box q \Rightarrow \langle \Rightarrow [\Rightarrow p \supset q] \rangle} \text{ } \quad \text{ } \\
 \frac{\quad}{\diamond p \supset \Box q \Rightarrow \Box(p \supset q)} \text{ } \quad \text{ }
 \end{array}$$

## More details are found in ...

The logic **FIK**, axiom system, and bi-nested calculus:

- Philippe Balbiani, G., Cigdem Gencer and Nicola Olivetti. "A Natural Intuitionistic Modal Logic: Axiomatization and Bi-Nested Calculus." In 32nd EACSL Annual Conference on Computer Science Logic (CSL 2024).

Shallow calculus and complexity:

- G. and Nicola Olivetti. "Taming Complexity in Intuitionistic Modal Logic: the case of FIK and its shallow calculus." In Advances in Modal Logic (AiML 2026), to appear.

Other IMLs under the bi-nested framework:

- G., Marianna Girlando, and Nicola Olivetti. 2026. "A Bi-Nested Calculus for Intuitionistic K: Proofs and Countermodels." Journal of Applied Non-Classical Logics, May, 1–40.
- Philippe Balbiani, G., Cigdem Gencer and Nicola Olivetti. "Local Intuitionistic Modal Logics and Their Calculi." In: Benzmüller, C., Heule, M.J., Schmidt, R.A. (eds) Automated Reasoning (IJCAR 2024).

**Thanks for your attention!**



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# From Bi-nested Sequents to Shallow Sequents

proof theory of the intuitionistic modal logic FIK

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Based on joint work with Philippe Balbiani, Çiğdem Gencer and Nicola Olivetti

LLAMA seminar @ ILLC, Amsterdam - 20<sup>th</sup> May, 2026