Cyclic Proofs, Hypersequents, and Transitive Closure Logic

Anupam Das*, Marianna Girlando⋆

* University of Birmingham ⋆ ILLC, Universtiy of Amsterdam

LLAMA Seminar, ILLC
Amsterdam, 21 September 2022

Outline

Outline

- The logics

Outline

- ▹ The logics
  - ★ Transitive Closure Logic (TCL)

Outline

- ▹ The logics
  - ★ Transitive Closure Logic (TCL)
  - ★ Identity-free Propositional Dynamic Logic (PDL^{+})

Outline

- The logics
  - Transitive Closure Logic (TCL)
  - Identity-free Propositional Dynamic Logic (PDL$^+$)

- Cyclic sequent proofs

Outline

- The logics
  - Transitive Closure Logic (TCL)
  - Identity-free Propositional Dynamic Logic (PDL^+)

- Cyclic sequent proofs
  - Cyclic sequents for PDL^+ [Niwiński & Walukiewicz, 1996]

Outline

- The logics
  - Transitive Closure Logic (TCL)
  - Identity-free Propositional Dynamic Logic (PDL^+)

- Cyclic sequent proofs
  - Cyclic sequents for PDL^+ [Niwiński & Walukiewicz, 1996]
  - Cyclic sequents for TCL [Cohen & Rowe, 2018, 2020]

Outline

- The logics
  - Transitive Closure Logic (TCL)
  - Identity-free Propositional Dynamic Logic (PDL^+)
- Cyclic sequent proofs
  - Cyclic sequents for PDL^+ [Niwiński & Walukiewicz, 1996]
  - Cyclic sequents for TCL [Cohen & Rowe, 2018, 2020]
- A simulation problem Cut-free PDL^+ proofs cannot be simulated by cut-free sequent proofs for TCL

Outline

- The logics
  - ★ Transitive Closure Logic (TCL)
  - ★ Identity-free Propositional Dynamic Logic (PDL^+)
- Cyclic sequent proofs
  - ★ Cyclic sequents for PDL^+ [Niwiński & Walukiewicz, 1996]
  - ★ Cyclic sequents for TCL [Cohen & Rowe, 2018, 2020]
- A simulation problem Cut-free PDL^+ proofs cannot be simulated by cut-free sequent proofs for TCL
- Cyclic hypersequents for TCL

Outline

- The logics
  - Transitive Closure Logic (TCL)
  - Identity-free Propositional Dynamic Logic (PDL^+)
- Cyclic sequent proofs
  - Cyclic sequents for PDL^+ [Niwiński & Walukiewicz, 1996]
  - Cyclic sequents for TCL [Cohen & Rowe, 2018, 2020]
- A simulation problem Cut-free PDL^+ proofs cannot be simulated by cut-free sequent proofs for TCL
- Cyclic hypersequents for TCL
  - Soundness of HTC

Outline

- The logics
  - Transitive Closure Logic (TCL)
  - Identity-free Propositional Dynamic Logic (PDL^{+})

- Cyclic sequent proofs
  - Cyclic sequents for PDL^{+} [Niwiński & Walukiewicz, 1996]
  - Cyclic sequents for TCL [Cohen & Rowe, 2018, 2020]

- A simulation problem Cut-free PDL^{+} proofs cannot be simulated by cut-free sequent proofs for TCL

- Cyclic hypersequents for TCL
  - Soundness of HTC
  - Cut-free PDL^{+} proofs can be simulated by cut-free hypersequent proofs for TCL

The logics

Transitive Closure Logic TCL

- [Immerman, 1987], [Gurevich, 1988], [Grädel, 1991], . . . , [Avron, 2003]

Transitive Closure Logic TCL

- [Immerman, 1987], [Gurevich, 1988], [Grädel, 1991], . . . , [Avron, 2003]

- Intermediate between first and second order logic

Transitive Closure Logic TCL

- [Immerman, 1987], [Gurevich, 1988], [Grädel, 1991], . . . , [Avron, 2003]
- Intermediate between first and second order logic
- Expresses the transitive closure of any given relation (unbounded reachability)

Transitive Closure Logic TCL

- [Immerman, 1987], [Gurevich, 1988], [Grädel, 1991], . . . , [Avron, 2003]
- Intermediate between first and second order logic
- Expresses the transitive closure of any given relation (unbounded reachability)
- Expressive enough to capture inductive principles

Transitive Closure Logic TCL

- [Immerman, 1987], [Gurevich, 1988], [Grädel, 1991], . . . , [Avron, 2003]
- Intermediate between first and second order logic
- Expresses the transitive closure of any given relation (unbounded reachability)
- Expressive enough to capture inductive principles
- "Simple" enough to be suitable for automation

Transitive Closure Logic TCL

- [Immerman, 1987], [Gurevich, 1988], [Grädel, 1991], . . . , [Avron, 2003]
- Intermediate between first and second order logic
- Expresses the transitive closure of any given relation (unbounded reachability)
- Expressive enough to capture inductive principles
- “Simple” enough to be suitable for automation
- Interprets Kleene Algebra and PDL

Transitive Closure Logic TCL, formally

Transitive Closure Logic TCL, formally

$$t \quad ::= \quad x \mid f(t_1, \ldots, t_n)$$

Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \ldots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \end{array}$$

Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \ldots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \ldots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

$\mathrm{TC}(A)(s,t)$ $\overline{\mathrm{TC}}(A)(s,t)$

Transitive Closure Logic TCL, formally

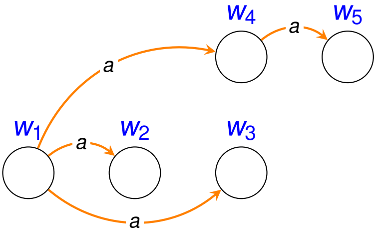
$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \ldots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

$$\overline{\mathrm{TC}(A)(s,t)} := \overline{\mathrm{TC}}(\bar{A})(s,t) \quad \overline{\overline{\mathrm{TC}}(A)(s,t)} := \mathrm{TC}(\bar{A})(s,t)$$

Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \ldots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

$$\overline{\mathrm{TC}(A)(s,t)} := \overline{\mathrm{TC}}(\bar{A})(s,t) \quad \overline{\overline{\mathrm{TC}}(A)(s,t)} := \mathrm{TC}(\bar{A})(s,t)$$



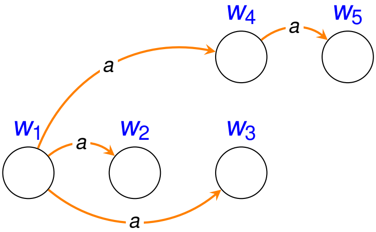
Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \dots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

$$\overline{\mathrm{TC}(A)(s,t)} := \overline{\mathrm{TC}}(\bar{A})(s,t) \quad \overline{\overline{\mathrm{TC}}(A)(s,t)} := \mathrm{TC}(\bar{A})(s,t)$$

$\rho(s) = w_1$

$\rho(t) = w_5$



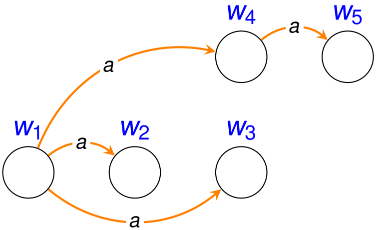
Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \dots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

$$\overline{\mathrm{TC}(A)(s,t)} := \overline{\mathrm{TC}}(\bar{A})(s,t) \quad \overline{\overline{\mathrm{TC}}(A)(s,t)} := \mathrm{TC}(\bar{A})(s,t)$$

$\rho(s) = w_1$

$\rho(t) = w_5$



$\mathcal{M}, \rho \models \mathrm{TC}(a)(s,t) \rightsquigarrow$ “There is a finite (> 0) A-path between $\rho(s)$ and $\rho(t)$”

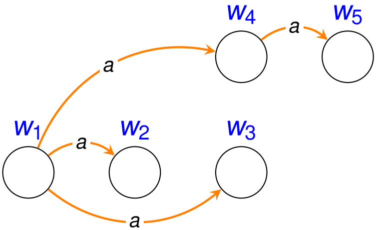
Transitive Closure Logic TCL, formally

$$\begin{array}{rcl} t & ::= & x \mid f(t_1, \ldots, t_n) \\ A, B & ::= & p(t) \mid \bar{p}(t) \mid a(s,t) \mid \bar{a}(s,t) \\ & & \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \\ & & \mid \mathrm{TC}(\lambda x, y.A)(s,t) \mid \overline{\mathrm{TC}}(\lambda x, y.A)(s,t) \end{array}$$

$$\overline{\mathrm{TC}(A)(s,t)} := \overline{\mathrm{TC}}(\bar{A})(s,t) \quad \overline{\overline{\mathrm{TC}}(A)(s,t)} := \mathrm{TC}(\bar{A})(s,t)$$

$\rho(s) = w_1$

$\rho(t) = w_5$



$\mathcal{M}, \rho \models \mathrm{TC}(a)(s,t) \rightsquigarrow$ "There is a finite (> 0) A-path between $\rho(s)$ and $\rho(t)$"

$$\mathrm{TC}(A)(s,t) \iff A(s,t) \vee \exists z(A(s,z) \wedge \mathrm{TC}(A)(z,t))$$

$$\overline{\mathrm{TC}}(A)(s,t) \iff A(s,t) \wedge \forall z(A(s,z) \vee \overline{\mathrm{TC}}(A)(z,t))$$

Identity-free Propositional Dynamic Logic PDL (PDL^{+})

Identity-free Propositional Dynamic Logic PDL (PDL^+)

$$A, B \quad ::= \quad p \mid \overline{p} \mid A \wedge B \mid A \vee B$$

Identity-free Propositional Dynamic Logic PDL (PDL^{+})

$$A, B \quad ::= \quad p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A$$

Identity-free Propositional Dynamic Logic PDL (PDL$^+$)

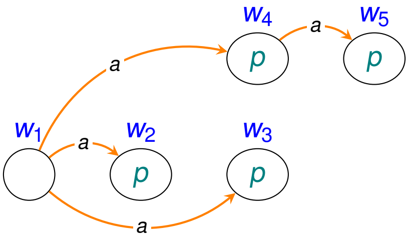
$$\mathcal{A} = \{a, b, c, \dots\}$$

$$\begin{array}{rcl} A, B & ::= & p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A \\ \alpha, \beta & ::= & a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \end{array}$$

Identity-free Propositional Dynamic Logic PDL (PDL$^+$)

$$\mathcal{A} = \{a, b, c, \dots\}$$

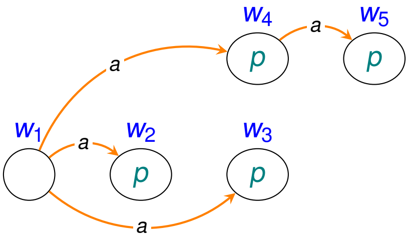
$$\begin{aligned} A, B \quad &::= \quad p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A \\ \alpha, \beta \quad &::= \quad a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \end{aligned}$$



Identity-free Propositional Dynamic Logic PDL (PDL^+)

$$\mathcal{A} = \{a, b, c, \dots\}$$

$$\begin{array}{rcl} A, B & ::= & p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A \\ \alpha, \beta & ::= & a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \end{array}$$

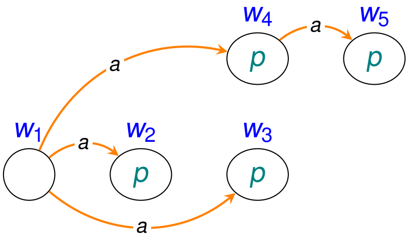


$w_1 \models [a^+]p$

Identity-free Propositional Dynamic Logic PDL (PDL$^+$)

$$\mathcal{A} = \{a, b, c, \ldots\}$$

$$\begin{array}{rcl} A, B & ::= & p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A \\ \alpha, \beta & ::= & a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \end{array}$$

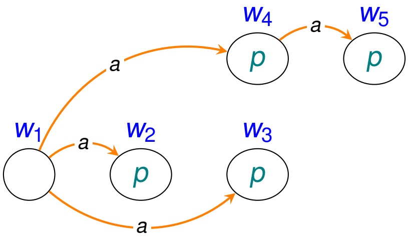


$w_1 \models [a^+]p$ iff for all w_i , if $a^+(w_1 w_i)$, then $w_i \models p$

Identity-free Propositional Dynamic Logic PDL (PDL$^+$)

$$\mathcal{A} = \{a, b, c, \dots\}$$

$$\begin{array}{rcl} A, B & ::= & p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A \\ \alpha, \beta & ::= & a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \end{array}$$



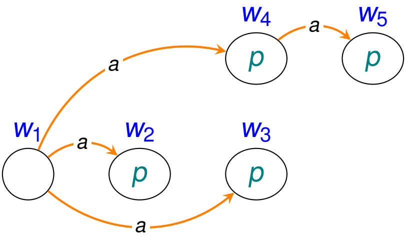
$w_1 \models [a^+]p$ iff for all w_i , if $a^+(w_1 w_i)$, then $w_i \models p$

$[a^+]p$ "all sequences ($>$ 0) of finite executions of a result in a p-state"

Identity-free Propositional Dynamic Logic PDL (PDL^+)

$$\mathcal{A} = \{a, b, c, \dots\}$$

$$\begin{aligned} A, B \quad &::= \quad p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid [\alpha]A \mid \langle\alpha\rangle A \\ \alpha, \beta \quad &::= \quad a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \end{aligned}$$



$$w_1 \models [a^+]p \quad \text{iff} \quad \text{for all } w_i \text{, if } a^+(w_1 w_i)\text{, then } w_i \models p$$

$[a^+]p$ “all sequences (> 0) of finite executions of a result in a p-state”

$$\langle\alpha^+\rangle A \iff \langle\alpha\rangle A \vee \langle\alpha\rangle\langle\alpha^+\rangle A \qquad [\alpha^+]A \iff [\alpha]A \wedge [\alpha][\alpha^+]A$$

TCL and PDL$^+$ (motivation)

TCL and PDL$^+$ (motivation)

$$[a]p \quad \rightsquigarrow \quad \mathrm{ST}([a]p)(x) := \forall y(a(x,y) \supset p(y))$$

TCL and PDL$^+$ (motivation)

$$[a]p \quad \rightsquigarrow \quad \mathrm{ST}([a]p)(x) := \forall y(a(x,y) \supset p(y))$$

$$[a^+]p \quad \rightsquigarrow \quad \mathrm{ST}([a^+]p)(x) := \forall y(\mathrm{TC}(a)(x,y) \supset p(y))$$

TCL and PDL^+ (motivation)

$$[a]p \quad \rightsquigarrow \quad \mathrm{ST}([a]p)(x) := \forall y(a(x,y) \supset p(y))$$

$$[a^+]p \quad \rightsquigarrow \quad \mathrm{ST}([a^+]p)(x) := \forall y(\mathrm{TC}(a)(x,y) \supset p(y))$$

Theorem For $\mathcal{M}$ relational model, w world and A modal formula,

$$\mathcal{M}, w \models A \quad \textit{iff} \quad \mathcal{M} \models \mathrm{ST}(A)(w)$$

TCL and PDL$^+$ (motivation)

$$[a]p \quad \rightsquigarrow \quad \mathrm{ST}([a]p)(x) := \forall y(a(x,y) \supset p(y))$$

$$[a^+]p \quad \rightsquigarrow \quad \mathrm{ST}([a^+]p)(x) := \forall y(\mathrm{TC}(a)(x,y) \supset p(y))$$

Theorem For $\mathcal{M}$ relational model, w world and A modal formula,

$$\mathcal{M}, w \models A \quad \textit{iff} \quad \mathcal{M} \models \mathrm{ST}(A)(w)$$

Can we lift this correspondence at the level of proofs?

TCL and PDL$^+$ (motivation)

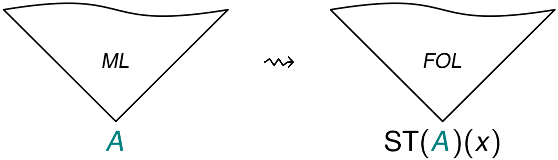
$$[a]p \quad \rightsquigarrow \quad \mathsf{ST}([a]p)(x) := \forall y(a(x,y) \supset p(y))$$

$$[a^+]p \quad \rightsquigarrow \quad \mathsf{ST}([a^+]p)(x) := \forall y(\mathsf{TC}(a)(x,y) \supset p(y))$$

Theorem For $\mathcal{M}$ relational model, w world and A modal formula,

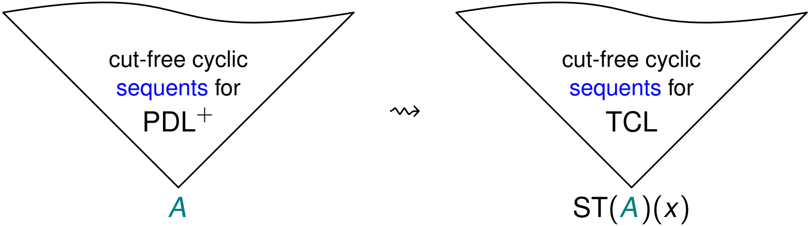
$$\mathcal{M}, w \models A \quad \textit{iff} \quad \mathcal{M} \models \mathsf{ST}(A)(w)$$

Can we lift this correspondence at the level of proofs?



Modal logic (ML), first order logic (FOL): [Miller & Volpe, 2015]

What about cyclic proofs?



cut-free cyclic sequents for PDL$^+$

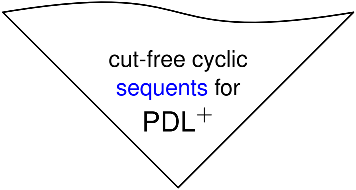
A

$\rightsquigarrow$

cut-free cyclic sequents for TCL

ST(A)(x)

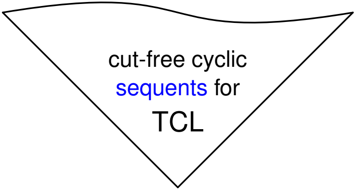
What about cyclic proofs?



cut-free cyclic sequents for PDL^{+}

A

$\rightsquigarrow$

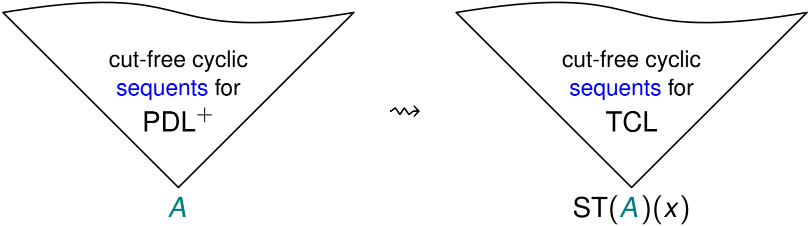


cut-free cyclic sequents for TCL

$\mathsf{ST}(A)(x)$

[Niwiński & Walukiewicz, 1996]
[Studer, 2008]
[Lange, 2003]

What about cyclic proofs?



cut-free cyclic sequents for PDL^+

A

$\rightsquigarrow$

cut-free cyclic sequents for TCL

$\mathrm{ST}(A)(x)$

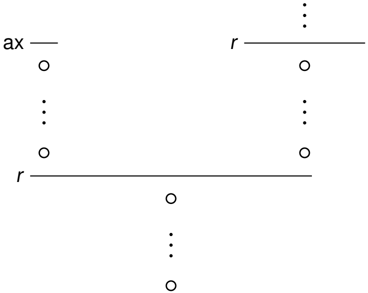
[Niwiński & Walukiewicz, 1996]
[Studer, 2008]
[Lange, 2003]

[Cohen & Rowe, 2018, 2020]

What are cyclic proofs?

What are cyclic proofs?

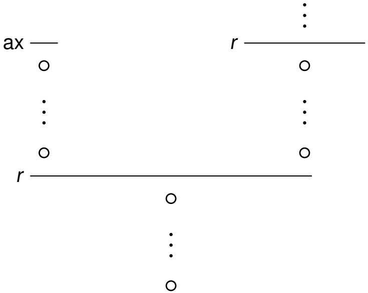
Cyclic proofs are possibly infinite binary trees, which:



What are cyclic proofs?

Cyclic proofs are possibly infinite binary trees, which:

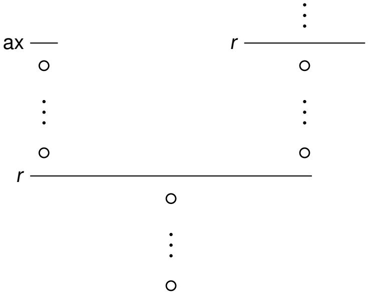
1. are regular



What are cyclic proofs?

Cyclic proofs are possibly infinite binary trees, which:

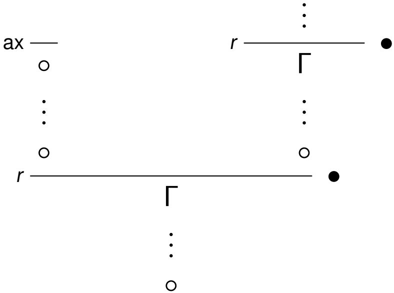
1. are regular
   - ⇝ contain a finite number of distinct subtrees



What are cyclic proofs?

Cyclic proofs are possibly infinite binary trees, which:

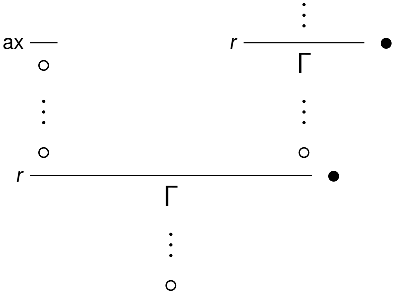
1. are regular
   - $\rightsquigarrow$ contain a finite number of distinct subtrees



What are cyclic proofs?

Cyclic proofs are possibly infinite binary trees, which:

1. are regular
   - ⇝ contain a finite number of distinct subtrees
2. satisfy a progress condition

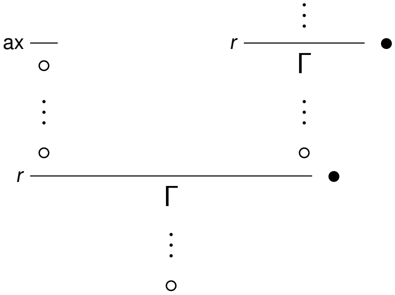


What are cyclic proofs?

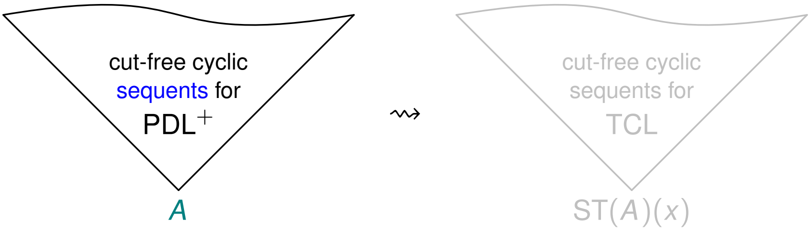
Cyclic proofs are possibly infinite binary trees, which:

1. are regular
   - ⇝ contain a finite number of distinct subtrees

2. satisfy a progress condition
   - ⇝ criterion to distinguish proofs from unsound infinite (regular) trees



Cyclic sequent proofs for PDL$^+$



cut-free cyclic sequents for PDL$^+$

A

$\rightsquigarrow$

cut-free cyclic sequents for TCL

$\mathrm{ST}(A)(x)$

[Niwiński & Walukiewicz, 1996]
[Studer, 2008]
[Lange, 2003]

[Cohen & Rowe, 2018, 2020]

GPD^{+} preproofs

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\}$$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad k\,\frac{\Gamma, A}{\langle a\rangle\Gamma, \Delta, [a]A}$$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad k\,\frac{\Gamma, A}{\langle a\rangle\Gamma, \Delta, [a]A}$$

$$\langle\cup\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\beta\rangle A}{\Gamma, \langle\alpha \cup \beta\rangle A} \qquad [\cup]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\beta]A}{\Gamma, [\alpha \cup \beta]A} \qquad \langle;\rangle\,\frac{\Gamma, \langle\alpha\rangle\langle\beta\rangle A}{\Gamma, \langle\alpha;\beta\rangle A} \qquad [;]\,\frac{\Gamma, [\alpha][\beta]A}{\Gamma, [\alpha;\beta]A}$$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad k\,\frac{\Gamma, A}{\langle a\rangle\Gamma, \Delta, [a]A}$$

$$\langle\cup\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\beta\rangle A}{\Gamma, \langle\alpha \cup \beta\rangle A} \qquad [\cup]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\beta]A}{\Gamma, [\alpha \cup \beta]A} \qquad \langle;\rangle\,\frac{\Gamma, \langle\alpha\rangle\langle\beta\rangle A}{\Gamma, \langle\alpha;\beta\rangle A} \qquad [;]\,\frac{\Gamma, [\alpha][\beta]A}{\Gamma, [\alpha;\beta]A}$$

$$\langle+\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\alpha\rangle\langle\alpha^+\rangle A}{\Gamma, \langle\alpha^+\rangle A} \qquad [+]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\alpha][\alpha^+]A}{\Gamma, [\alpha^+]A}$$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad k\,\frac{\Gamma, A}{\langle a\rangle\Gamma, \Delta, [a]A}$$

$$\langle\cup\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\beta\rangle A}{\Gamma, \langle\alpha \cup \beta\rangle A} \qquad [\cup]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\beta]A}{\Gamma, [\alpha \cup \beta]A} \qquad \langle;\rangle\,\frac{\Gamma, \langle\alpha\rangle\langle\beta\rangle A}{\Gamma, \langle\alpha;\beta\rangle A} \qquad [;]\,\frac{\Gamma, [\alpha][\beta]A}{\Gamma, [\alpha;\beta]A}$$

$$\langle+\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\alpha\rangle\langle\alpha^+\rangle A}{\Gamma, \langle\alpha^+\rangle A} \qquad [+]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\alpha][\alpha^+]A}{\Gamma, [\alpha^+]A}$$

$$\langle\alpha^+\rangle A \iff \langle\alpha\rangle A \vee \langle\alpha\rangle\langle\alpha^+\rangle A \qquad [\alpha^+]A \iff [\alpha]A \wedge [\alpha][\alpha^+]A$$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \quad \text{wk}\,\frac{\Gamma}{\Gamma, A} \quad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \quad k\,\frac{\Gamma, A}{\langle a\rangle\Gamma, \Delta, [a]A}$$

$$\langle\cup\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\beta\rangle A}{\Gamma, \langle\alpha \cup \beta\rangle A} \quad [\cup]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\beta]A}{\Gamma, [\alpha \cup \beta]A} \quad \langle;\rangle\,\frac{\Gamma, \langle\alpha\rangle\langle\beta\rangle A}{\Gamma, \langle\alpha;\beta\rangle A} \quad [;]\,\frac{\Gamma, [\alpha][\beta]A}{\Gamma, [\alpha;\beta]A}$$

$$\langle+\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\alpha\rangle\langle\alpha^+\rangle A}{\Gamma, \langle\alpha^+\rangle A} \qquad [+]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\alpha][\alpha^+]A}{\Gamma, [\alpha^+]A}$$

$$\langle\alpha^+\rangle A \iff \langle\alpha\rangle A \vee \langle\alpha\rangle\langle\alpha^+\rangle A \qquad [\alpha^+]A \iff [\alpha]A \wedge [\alpha][\alpha^+]A$$

Trace $\tau = (F_i)_{i<\omega}$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad k\,\frac{\Gamma, A}{\langle a\rangle\Gamma, \Delta, [a]A}$$

$$\langle\cup\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\beta\rangle A}{\Gamma, \langle\alpha\cup\beta\rangle A} \qquad [\cup]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\beta]A}{\Gamma, [\alpha\cup\beta]A} \qquad \langle;\rangle\,\frac{\Gamma, \langle\alpha\rangle\langle\beta\rangle A}{\Gamma, \langle\alpha;\beta\rangle A} \qquad [;]\,\frac{\Gamma, [\alpha][\beta]A}{\Gamma, [\alpha;\beta]A}$$

$$\langle+\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\alpha\rangle\langle\alpha^+\rangle A}{\Gamma, \langle\alpha^+\rangle A} \qquad [+]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\alpha][\alpha^+]A}{\Gamma, [\alpha^+]A}$$

$$\langle\alpha^+\rangle A \iff \langle\alpha\rangle A \vee \langle\alpha\rangle\langle\alpha^+\rangle A \qquad [\alpha^+]A \iff [\alpha]A \wedge [\alpha][\alpha^+]A$$

Trace $\tau = (F_i)_{i<\omega}$

Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $[\alpha^+]A$

GPD$^+$ preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{ax}\,\frac{}{\Gamma, p, \overline{p}} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \text{s}\vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{s}\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad k\,\frac{\Gamma, A}{\langle a \rangle \Gamma, \Delta, [a]A}$$

$$\langle\cup\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\beta\rangle A}{\Gamma, \langle\alpha \cup \beta\rangle A} \qquad [\cup]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\beta]A}{\Gamma, [\alpha \cup \beta]A} \qquad \langle;\rangle\,\frac{\Gamma, \langle\alpha\rangle\langle\beta\rangle A}{\Gamma, \langle\alpha;\beta\rangle A} \qquad [;]\,\frac{\Gamma, [\alpha][\beta]A}{\Gamma, [\alpha;\beta]A}$$

$$\langle+\rangle\,\frac{\Gamma, \langle\alpha\rangle A, \langle\alpha\rangle\langle\alpha^+\rangle A}{\Gamma, \langle\alpha^+\rangle A} \qquad [+]\,\frac{\Gamma, [\alpha]A \quad \Gamma, [\alpha][\alpha^+]A}{\Gamma, [\alpha^+]A}$$

$$\langle\alpha^+\rangle A \iff \langle\alpha\rangle A \vee \langle\alpha\rangle\langle\alpha^+\rangle A \qquad [\alpha^+]A \iff [\alpha]A \wedge [\alpha][\alpha^+]A$$

Trace $\tau = (F_i)_{i<\omega}$

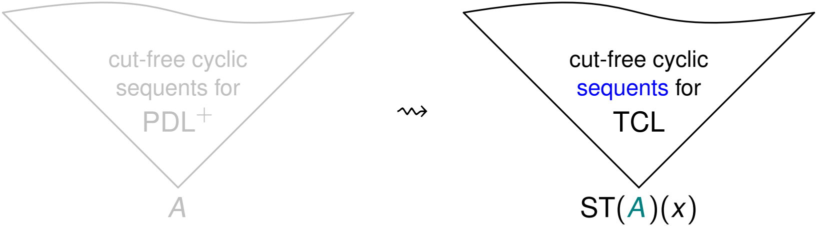
Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $[\alpha^+]A$

Progress condition: Every infinite branch has a progressing trace

Example

$$
{\scriptstyle\lor}\dfrac{{\scriptstyle\langle;\rangle}\dfrac{{\scriptstyle[+]}\dfrac{{\scriptstyle[\cup]}\dfrac{{\scriptstyle[;]}\dfrac{{\scriptstyle\langle+\rangle_0}\dfrac{{\scriptstyle k_a}\dfrac{{\scriptstyle\langle\cup\rangle}\dfrac{{\scriptstyle k_a}\dfrac{{\scriptstyle\text{init}}\dfrac{}{\overline{p}, p}}{[a]\overline{p}, \langle a\rangle p}}{[a]\overline{p}, \langle\beta\cup a\rangle p}}{[a][a]\overline{p}, \langle a\rangle\langle\beta\cup a\rangle p}}{[a][a]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}}{[aa]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}\quad{\scriptstyle 2[;]}\dfrac{{\scriptstyle\langle+\rangle_0}\dfrac{{\scriptstyle k_a}\dfrac{{\scriptstyle\langle\cup\rangle}\dfrac{{\scriptstyle\langle+\rangle_0}\dfrac{{\scriptstyle\langle;\rangle}\dfrac{{\scriptstyle k_b}\dfrac{{\scriptstyle\langle+\rangle_0}\dfrac{{\scriptstyle k_a}\dfrac{{\scriptstyle\text{init}}\dfrac{}{\overline{p}, p}}{[a]\overline{p}, \langle a\rangle p}}{[a]\overline{p}, \langle a^+\rangle p}}{[b][a]\overline{p}, \langle b\rangle\langle a^+\rangle p}}{[b][a]\overline{p}, \langle ba^+\rangle p}}{[b][a]\overline{p}, \langle\beta\rangle p}}{[b][a]\overline{p}, \langle\beta\cup a\rangle p}}{[a][b][a]\overline{p}, \langle a\rangle\langle\beta\cup a\rangle p}}{[a][b][a]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}}{[aba]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}}{[aa\cup aba]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}\quad{\scriptstyle[\cup]}\dfrac{{\scriptstyle[;]}\dfrac{{\scriptstyle\langle+\rangle_1}\dfrac{{\scriptstyle k_a}\dfrac{{\scriptstyle\langle+\rangle_1}\dfrac{{\scriptstyle k_a}\dfrac{\dfrac{\vdots}{[\alpha]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}\,\bullet}{[a][\alpha]\overline{p}, \langle a\rangle\langle a^+\rangle\langle\beta\cup a\rangle p}}{[a][\alpha]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}}{[a][a][\alpha]\overline{p}, \langle a\rangle\langle a^+\rangle\langle\beta\cup a\rangle p}}{[a][a][\alpha]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}}{[aa][\alpha]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}\quad\Gamma}{[aa\cup aba][\alpha]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}}{[(aa\cup aba)^+]\overline{p}, \langle a^+\rangle\langle\beta\cup a\rangle p}\,\bullet}{[(aa\cup aba)^+]\overline{p}, \langle a^+(\beta\cup a)\rangle p}}{[(aa\cup aba)^+]\overline{p}\lor\langle a^+((ba^+)^+\cup a)\rangle p}
$$

$$\alpha = (aa\cup aba)^+ \qquad \beta = (ba^+)^+$$



cut-free cyclic sequents for PDL^+

A

$\rightsquigarrow$

cut-free cyclic sequents for TCL

$\mathrm{ST}(A)(x)$

[Niwiński & Walukiewicz, 1996]
[Studer, 2008]
[Lange, 2003]

[Cohen & Rowe, 2018, 2020]

GTC preproofs

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\}$$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{id}\ \frac{}{\Gamma, p(t), \bar{p}(t)} \qquad \text{wk}\ \frac{\Gamma}{\Gamma, \Gamma'} \qquad \sigma\ \frac{\Gamma}{\sigma(\Gamma)} \qquad \text{cut}\ \frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

$$\vee_i\ \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\ i \in \{0, 1\} \qquad \wedge\ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \exists\ \frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \qquad \forall\ \frac{\Gamma, A[c/x]}{\Gamma, \forall x A}\ c \text{ fresh}$$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{id}\,\frac{}{\Gamma, p(t), \bar{p}(t)} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, \Gamma'} \qquad \sigma\,\frac{\Gamma}{\sigma(\Gamma)} \qquad \text{cut}\,\frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

$$\vee_i\,\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\; i \in \{0,1\} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \exists\,\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \qquad \forall\,\frac{\Gamma, A[c/x]}{\Gamma, \forall x A}\; c \text{ fresh}$$

$$TC_0\,\frac{\Gamma, A(s,t)}{\Gamma, \mathrm{TC}(A)(s,t)} \qquad TC_1\,\frac{\Gamma, A(s,r) \quad \Gamma, \mathrm{TC}(A)(r,t)}{\Gamma, \mathrm{TC}(A)(s,t)}$$

$$\overline{TC}\,\frac{\Gamma, A(s,t) \quad \Gamma, A(s,c), \overline{\mathrm{TC}}(A)(c,t)}{\Gamma, \overline{\mathrm{TC}}(A)(s,t)}\; c \text{ fresh}$$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{id}\ \frac{}{\Gamma, p(t), \bar{p}(t)} \qquad \text{wk}\ \frac{\Gamma}{\Gamma, \Gamma'} \qquad \sigma\ \frac{\Gamma}{\sigma(\Gamma)} \qquad \text{cut}\ \frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

$$\vee_i\ \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\ i \in \{0,1\} \qquad \wedge\ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \exists\ \frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \qquad \forall\ \frac{\Gamma, A[c/x]}{\Gamma, \forall x A}\ c \text{ fresh}$$

$$TC_0\ \frac{\Gamma, A(s,t)}{\Gamma, \mathrm{TC}(A)(s,t)} \qquad TC_1\ \frac{\Gamma, A(s,r) \quad \Gamma, \mathrm{TC}(A)(r,t)}{\Gamma, \mathrm{TC}(A)(s,t)}$$

$$\overline{TC}\ \frac{\Gamma, A(s,t) \quad \Gamma, A(s,c), \overline{\mathrm{TC}}(A)(c,t)}{\Gamma, \overline{\mathrm{TC}}(A)(s,t)}\ c \text{ fresh}$$

$$\begin{aligned}
\mathrm{TC}(A)(s,t) \quad &\Longleftrightarrow \quad A(s,t) \vee \exists z (A(s,z) \wedge \mathrm{TC}(A)(z,t)) \\
\overline{\mathrm{TC}}(A)(s,t) \quad &\Longleftrightarrow \quad A(s,t) \wedge \forall z (A(s,z) \vee \overline{\mathrm{TC}}(A)(z,t))
\end{aligned}$$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{id}\,\frac{}{\Gamma, p(t), \bar{p}(t)} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, \Gamma'} \qquad \sigma\,\frac{\Gamma}{\sigma(\Gamma)} \qquad \text{cut}\,\frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

$$\vee_i\,\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\, i \in \{0,1\} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \exists\,\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \qquad \forall\,\frac{\Gamma, A[c/x]}{\Gamma, \forall x A}\, c \text{ fresh}$$

$$TC_0\,\frac{\Gamma, A(s,t)}{\Gamma, \mathrm{TC}(A)(s,t)} \qquad TC_1\,\frac{\Gamma, A(s,r) \quad \Gamma, \mathrm{TC}(A)(r,t)}{\Gamma, \mathrm{TC}(A)(s,t)}$$

$$\overline{TC}\,\frac{\Gamma, A(s,t) \quad \Gamma, A(s,c), \overline{\mathrm{TC}}(A)(c,t)}{\Gamma, \overline{\mathrm{TC}}(A)(s,t)}\, c \text{ fresh}$$

$$\mathrm{TC}(A)(s,t) \iff A(s,t) \vee \exists z (A(s,z) \wedge \mathrm{TC}(A)(z,t))$$
$$\overline{\mathrm{TC}}(A)(s,t) \iff A(s,t) \wedge \forall z (A(s,z) \vee \overline{\mathrm{TC}}(A)(z,t))$$

Trace $\tau = (F_i)_{i<\omega}$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{id}\,\frac{}{\Gamma, p(t), \bar{p}(t)} \qquad \text{wk}\,\frac{\Gamma}{\Gamma, \Gamma'} \qquad \sigma\,\frac{\Gamma}{\sigma(\Gamma)} \qquad \text{cut}\,\frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

$$\vee_i\,\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\ i \in \{0,1\} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \exists\,\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \qquad \forall\,\frac{\Gamma, A[c/x]}{\Gamma, \forall x A}\ c \text{ fresh}$$

$$TC_0\,\frac{\Gamma, A(s,t)}{\Gamma, \mathrm{TC}(A)(s,t)} \qquad TC_1\,\frac{\Gamma, A(s,r) \quad \Gamma, \mathrm{TC}(A)(r,t)}{\Gamma, \mathrm{TC}(A)(s,t)}$$

$$\overline{TC}\,\frac{\Gamma, A(s,t) \quad \Gamma, A(s,c), \overline{\mathrm{TC}}(A)(c,t)}{\Gamma, \overline{\mathrm{TC}}(A)(s,t)}\ c \text{ fresh}$$

$$\begin{aligned} \mathrm{TC}(A)(s,t) &\iff A(s,t) \vee \exists z(A(s,z) \wedge \mathrm{TC}(A)(z,t)) \\ \overline{\mathrm{TC}}(A)(s,t) &\iff A(s,t) \wedge \forall z(A(s,z) \vee \overline{\mathrm{TC}}(A)(z,t)) \end{aligned}$$

Trace $\tau = (F_i)_{i<\omega}$

Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $\overline{\mathrm{TC}}(A)(\cdot,\cdot)$

GTC preproofs

$$\Gamma = \{A_1, \ldots, A_k\} \qquad fm(\Gamma) = A_1 \vee \ldots \vee A_k$$

$$\text{id}\,\dfrac{}{\Gamma, p(t), \bar{p}(t)} \qquad \text{wk}\,\dfrac{\Gamma}{\Gamma, \Gamma'} \qquad \sigma\,\dfrac{\Gamma}{\sigma(\Gamma)} \qquad \text{cut}\,\dfrac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

$$\vee_i\,\dfrac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\; i \in \{0,1\} \qquad \wedge\,\dfrac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \exists\,\dfrac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \qquad \forall\,\dfrac{\Gamma, A[c/x]}{\Gamma, \forall x A}\; c \text{ fresh}$$

$$TC_0\,\dfrac{\Gamma, A(s,t)}{\Gamma, \mathrm{TC}(A)(s,t)} \qquad TC_1\,\dfrac{\Gamma, A(s,r) \quad \Gamma, \mathrm{TC}(A)(r,t)}{\Gamma, \mathrm{TC}(A)(s,t)}$$

$$\overline{TC}\,\dfrac{\Gamma, A(s,t) \quad \Gamma, A(s,c), \overline{\mathrm{TC}}(A)(c,t)}{\Gamma, \overline{\mathrm{TC}}(A)(s,t)}\; c \text{ fresh}$$

$$\begin{aligned} \mathrm{TC}(A)(s,t) &\iff A(s,t) \vee \exists z(A(s,z) \wedge \mathrm{TC}(A)(z,t)) \\ \overline{\mathrm{TC}}(A)(s,t) &\iff A(s,t) \wedge \forall z(A(s,z) \vee \overline{\mathrm{TC}}(A)(z,t)) \end{aligned}$$

Trace $\tau = (F_i)_{i<\omega}$

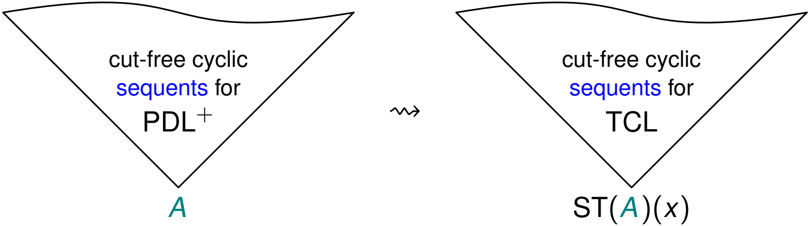
Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $\overline{\mathrm{TC}}(A)(\cdot,\cdot)$

Progress condition: Every infinite branch has a progressing trace

Example

$$
{\scriptstyle \vee}\dfrac{{\scriptstyle \overline{TC}}\dfrac{{\scriptstyle TC_0}\dfrac{{\scriptstyle \mathrm{id}}\dfrac{}{a(c,d),\bar{a}(c,d)}}{a(c,d),\mathrm{TC}(\bar{a})(c,d)} \qquad {\scriptstyle TC_1}\dfrac{{\scriptstyle \mathrm{id}}\dfrac{}{a(c,e),\bar{a}(c,e)} \qquad {\scriptstyle \overline{TC}}\dfrac{\vdots}{\overline{\mathrm{TC}}(a)(e,d),\mathrm{TC}(\bar{a})(e,d)}\ \bullet}{a(c,e),\overline{\mathrm{TC}}(a)(e,d),\mathrm{TC}(\bar{a})(c,d)}}{\overline{\mathrm{TC}}(a)(c,d),\mathrm{TC}(\bar{a})(c,d)}\ \bullet}{\overline{\mathrm{TC}}(a)(c,d)\vee\mathrm{TC}(\bar{a})(c,d)}
$$

What about simulation?



cut-free cyclic sequents for PDL^{+}

A

$\rightsquigarrow$

cut-free cyclic sequents for TCL

$\mathsf{ST}(A)(x)$

[Niwiński & Walukiewicz, 1996]
[Studer, 2008]
[Lange, 2003]

[Cohen & Rowe, 2018, 2020]

No..

No..

Theorem (Incompleteness) There is a valid PDL$^+$ formula A which has a cut-free sequent proof, but such that ST(A)(x) has no cut-free TCL cyclic sequent proof.

$$A = \langle (aa \cup aba)^+ \rangle p \supset \langle a^+((ba^+)^+ \cup a) \rangle p$$

No..

Theorem (Incompleteness) There is a valid PDL$^+$ formula A which has a cut-free sequent proof, but such that ST(A)(x) has no cut-free TCL cyclic sequent proof.

$$A \;=\; \langle (aa \cup aba)^+ \rangle p \supset \langle a^+((ba^+)^+ \cup a) \rangle p$$

Corollary The class of cyclic sequent proofs for TCL does not enjoy cut-admissibility.

$$\text{cut}\,\frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}$$

Why simulation fails?

Why simulation fails?

$$\mathrm{ST}(\langle a\rangle p)(x) \qquad := \qquad \exists y(a(x, y) \wedge p(y))$$

Why simulation fails?

$$
\begin{aligned}
\mathrm{ST}(\langle a \rangle p)(x) &:= \exists y(a(x, y) \wedge p(y)) \\
\mathrm{ST}(\langle a_0 \cup a_1 \rangle p)(x) &:= \exists y((a_0(x, y) \vee a_1(x, y)) \wedge p(y))
\end{aligned}
$$

Why simulation fails?

$$\begin{aligned} \mathrm{ST}(\langle a\rangle p)(x) &\quad:=\quad \exists y(a(x,y) \wedge p(y)) \\ \mathrm{ST}(\langle a_0 \cup a_1\rangle p)(x) &\quad:=\quad \exists y((a_0(x,y) \vee a_1(x,y)) \wedge p(y)) \end{aligned}$$

$$\langle\cup\rangle_i \, \frac{\langle a_i\rangle p}{\langle a_0 \cup a_1\rangle p} \; i \in \{0,1\}$$

Why simulation fails?

$$\begin{array}{rcl}
\mathrm{ST}(\langle a\rangle p)(x) & := & \exists y(a(x,y)\wedge p(y)) \\
\mathrm{ST}(\langle a_0\cup a_1\rangle p)(x) & := & \exists y((a_0(x,y)\vee a_1(x,y))\wedge p(y))
\end{array}$$

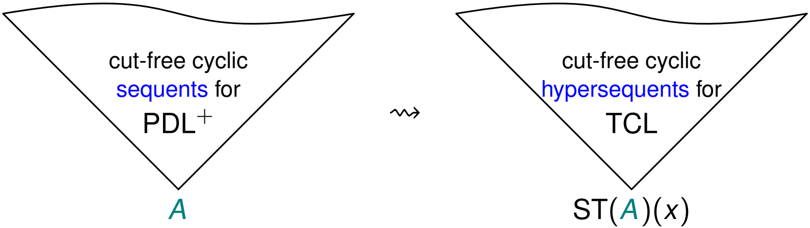
$$\langle\cup\rangle_i\,\frac{\langle a_i\rangle p}{\langle a_0\cup a_1\rangle p}\ i\in\{0,1\} \quad\rightsquigarrow\quad \begin{array}{c}
\mathrm{ST}(\langle a_i\rangle p)(x) \\ \hdashline
\exists y(a_i(x,y)\wedge p(y)) \\
\vdots \\
\\
\vdots \\
\exists y((a_0(x,y)\vee a_1(x,y))\wedge p(y)) \\ \hdashline
\mathrm{ST}(\langle a_0\cup a_1\rangle p)(x)
\end{array}$$

Why simulation fails?

$$
\begin{aligned}
\mathrm{ST}(\langle a \rangle p)(x) \quad &:= \quad \exists y(a(x, y) \land p(y)) \\
\mathrm{ST}(\langle a_0 \cup a_1 \rangle p)(x) \quad &:= \quad \exists y((a_0(x, y) \lor a_1(x, y)) \land p(y))
\end{aligned}
$$

$$
{\scriptstyle \langle \cup \rangle_i}\, \frac{\langle a_i \rangle p}{\langle a_0 \cup a_1 \rangle p}\; i \in \{0, 1\} \qquad \rightsquigarrow \qquad
\begin{array}{c}
\mathrm{ST}(\langle a_i \rangle p)(x) \\
\hdashline
\exists y(a_i(x, y) \land p(y)) \\
\vdots \\
\textcircled{?} \\
\vdots \\
\exists y((a_0(x, y) \lor a_1(x, y)) \land p(y)) \\
\hdashline
\mathrm{ST}(\langle a_0 \cup a_1 \rangle p)(x)
\end{array}
$$

Solution: more structure!



cut-free cyclic
sequents for
PDL$^+$

A

$\rightsquigarrow$

cut-free cyclic
hypersequents for
TCL

$\mathrm{ST}(A)(x)$

[Niwiński & Walukiewicz, 1996]
[Studer, 2008]
[Lange, 2003]

Annotated hypersequents for TCL: HTC

Annotated hypersequents for TCL: HTC

Annotated cedent

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

Annotated hypersequents for TCL: HTC

Annotated cedent (**x** set of variables)

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

Annotated hypersequents for TCL: HTC

Annotated cedent (**x** set of variables)

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

$$fm(\{A_1, \ldots, A_k\}^{\mathbf{x}}) \quad := \quad \exists \mathbf{x}(A_1 \wedge \ldots \wedge A_k)$$

Annotated hypersequents for TCL: HTC

Annotated cedent (**x** set of variables)

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

$$
\begin{aligned}
fm(\{A_1, \ldots, A_k\}^{\mathbf{x}}) \quad &:= \quad \exists \mathbf{x}(A_1 \wedge \ldots \wedge A_k) \\
fm(\{A_1, \ldots, A_k\}^{\emptyset}) \quad &:= \quad A_1 \wedge \ldots \wedge A_k
\end{aligned}
$$

Annotated hypersequents for TCL: HTC

Annotated cedent (**x** set of variables)

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

$$\begin{array}{lcl} fm(\{A_1, \ldots, A_k\}^{\mathbf{x}}) & := & \exists \mathbf{x}(A_1 \wedge \ldots \wedge A_k) \\ fm(\{A_1, \ldots, A_k\}^{\varnothing}) & := & A_1 \wedge \ldots \wedge A_k \\ fm(\{\,\}^{\varnothing}) & := & \top \end{array}$$

Annotated hypersequents for TCL: HTC

Annotated cedent (**x** set of variables)

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

$$\begin{aligned} fm(\{A_1, \ldots, A_k\}^{\mathbf{x}}) &:= \exists \mathbf{x}(A_1 \wedge \ldots \wedge A_k) \\ fm(\{A_1, \ldots, A_k\}^{\varnothing}) &:= A_1 \wedge \ldots \wedge A_k \\ fm(\{\,\}^{\varnothing}) &:= \top \end{aligned}$$

Annotated hypersequent (set of cedents)

$$\mathbf{S} = \{\Gamma_1\}^{\mathbf{x}_1}, \ldots, \{\Gamma_n\}^{\mathbf{x}_n}$$

Annotated hypersequents for TCL: HTC

Annotated cedent ($\mathbf{x}$ set of variables)

$$\{A_1, \ldots, A_k\}^{\mathbf{x}}$$

$$\begin{array}{lcc} fm(\{A_1, \ldots, A_k\}^{\mathbf{x}}) & := & \exists \mathbf{x}(A_1 \wedge \ldots \wedge A_k) \\ fm(\{A_1, \ldots, A_k\}^{\emptyset}) & := & A_1 \wedge \ldots \wedge A_k \\ fm(\{\,\}^{\emptyset}) & := & \top \end{array}$$

Annotated hypersequent (set of cedents)

$$\mathbf{S} = \{\Gamma_1\}^{\mathbf{x}_1}, \ldots, \{\Gamma_n\}^{\mathbf{x}_n}$$

$$fm(\{\Gamma_1\}^{\mathbf{x}_1}, \ldots, \{\Gamma_n\}^{\mathbf{x}_n}) := fm(\{\Gamma_1\}^{\mathbf{x}_1}) \vee \ldots \vee fm(\{\Gamma_n\}^{\mathbf{x}_n})$$

HTC preproofs

HTC preproofs

$$\text{init}\,\frac{}{\{\,\}^{\varnothing}} \qquad \text{wk}\,\frac{\mathbf{S}}{\mathbf{S}, \mathbf{S}'}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\,\}^{\varnothing}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S}, \mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A\}^{\mathbf{x}}, \{\overline{A}\}^{\varnothing}}\ A \text{ closed}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\ \}^{\varnothing}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S}, \mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A\}^{\mathbf{x}}, \{\overline{A}\}^{\varnothing}}\ A \text{ closed}$$

$$\cup\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}} \quad \mathbf{S}, \{\Delta\}^{\mathbf{y}}}{\mathbf{S}, \{\Gamma, \Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{array}{l}\text{fv}(\Delta) \cap \mathbf{x} = \emptyset \\ \text{fv}(\Gamma) \cap \mathbf{y} = \emptyset\end{array} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma, A, B\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A \wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S}, \{\Gamma, A_i\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A_0 \vee A_1\}^{\mathbf{x}}}\ i \in \{0, 1\}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\ \}^{\varnothing}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S}, \mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A\}^{\mathbf{x}}, \{\overline{A}\}^{\varnothing}}\ A \text{ closed}$$

$$\cup\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}} \quad \mathbf{S}, \{\Delta\}^{\mathbf{y}}}{\mathbf{S}, \{\Gamma, \Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{array}{l}\mathrm{fv}(\Delta) \cap \mathbf{x} = \emptyset \\ \mathrm{fv}(\Gamma) \cap \mathbf{y} = \emptyset\end{array} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma, A, B\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A \wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S}, \{\Gamma, A_i\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A_0 \vee A_1\}^{\mathbf{x}}}\ i \in \{0, 1\}$$

$$\exists\ \frac{\mathbf{S}, \{\Gamma, A(y)\}^{\mathbf{x},y}}{\mathbf{S}, \{\Gamma, \exists z(A(z))\}^{\mathbf{x}}}\ y \text{ fresh} \qquad \text{inst}\ \frac{\mathbf{S}, \{\Gamma(t)\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma(y)\}^{\mathbf{x},y}}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\,\}^{\varnothing}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S}, \mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A\}^{\mathbf{x}}, \{\overline{A}\}^{\varnothing}}\ A \text{ closed}$$

$$\cup\ \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}} \quad \mathbf{S}, \{\Delta\}^{\mathbf{y}}}{\mathbf{S}, \{\Gamma, \Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{matrix}\mathrm{fv}(\Delta) \cap \mathbf{x} = \emptyset \\ \mathrm{fv}(\Gamma) \cap \mathbf{y} = \emptyset\end{matrix} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma, A, B\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A \wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S}, \{\Gamma, A_i\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A_0 \vee A_1\}^{\mathbf{x}}}\ i \in \{0, 1\}$$

$$\forall\ \frac{\mathbf{S}, \{\Gamma, A(f(\mathbf{x}))\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, \forall z(A(z))\}^{\mathbf{x}}}\ f \text{ fresh} \qquad \exists\ \frac{\mathbf{S}, \{\Gamma, A(y)\}^{\mathbf{x},y}}{\mathbf{S}, \{\Gamma, \exists z(A(z))\}^{\mathbf{x}}}\ y \text{ fresh} \qquad \text{inst}\ \frac{\mathbf{S}, \{\Gamma(t)\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma(y)\}^{\mathbf{x},y}}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\,\}^{\varnothing}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S},\mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\}^{\mathbf{x}},\{\overline{A}\}^{\varnothing}}\ A \text{ closed}$$

$$\cup\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}} \quad \mathbf{S},\{\Delta\}^{\mathbf{y}}}{\mathbf{S},\{\Gamma,\Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{array}{l}\mathrm{fv}(\Delta)\cap\mathbf{x}=\emptyset\\ \mathrm{fv}(\Gamma)\cap\mathbf{y}=\emptyset\end{array} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma,A,B\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S},\{\Gamma,A_i\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A_0\vee A_1\}^{\mathbf{x}}}\ i\in\{0,1\}$$

$$\forall\ \frac{\mathbf{S},\{\Gamma,A(f(\mathbf{x}))\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\forall z(A(z))\}^{\mathbf{x}}}\ f \text{ fresh} \qquad \exists\ \frac{\mathbf{S},\{\Gamma,A(y)\}^{\mathbf{x},y}}{\mathbf{S},\{\Gamma,\exists z(A(z))\}^{\mathbf{x}}}\ y \text{ fresh} \qquad \text{inst}\ \frac{\mathbf{S},\{\Gamma(t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma(y)\}^{\mathbf{x},y}}$$

$$\text{tc}\ \frac{\mathbf{S},\{\Gamma,A(s,t)\}^{\mathbf{x}},\{\Gamma,A(s,z),\mathrm{TC}(A)(z,t)\}^{\mathbf{x},z}}{\mathbf{S},\{\Gamma,\mathrm{TC}(A)(s,t)\}^{\mathbf{x}}}\ z \text{ fresh}$$

$$\overline{\text{tc}}\ \frac{\mathbf{S},\{\Gamma,A(s,t),A(s,f(\mathbf{x}))\}^{\mathbf{x}},\{\Gamma,A(s,t),\overline{\mathrm{TC}}(A)(f(\mathbf{x}),t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\overline{\mathrm{TC}}(A)(s,t)\}^{\mathbf{x}}}\ f \text{ fresh}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\,\}^{\varnothing}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S},\mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\}^{\mathbf{x}},\{\overline{A}\}^{\varnothing}}\ A\ \text{closed}$$

$$\cup\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}} \qquad \mathbf{S},\{\Delta\}^{\mathbf{y}}}{\mathbf{S},\{\Gamma,\Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{array}{l}\mathrm{fv}(\Delta)\cap\mathbf{x}=\emptyset\\ \mathrm{fv}(\Gamma)\cap\mathbf{y}=\emptyset\end{array} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma,A,B\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S},\{\Gamma,A_i\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A_0\vee A_1\}^{\mathbf{x}}}\ i\in\{0,1\}$$

$$\forall\ \frac{\mathbf{S},\{\Gamma,A(f(\mathbf{x}))\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\forall z(A(z))\}^{\mathbf{x}}}\ f\ \text{fresh} \qquad \exists\ \frac{\mathbf{S},\{\Gamma,A(y)\}^{\mathbf{x},y}}{\mathbf{S},\{\Gamma,\exists z(A(z))\}^{\mathbf{x}}}\ y\ \text{fresh} \qquad \text{inst}\ \frac{\mathbf{S},\{\Gamma(t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma(y)\}^{\mathbf{x},y}}$$

$$\text{tc}\ \frac{\mathbf{S},\{\Gamma,A(s,t)\}^{\mathbf{x}},\{\Gamma,A(s,z),\mathrm{TC}(A)(z,t)\}^{\mathbf{x},z}}{\mathbf{S},\{\Gamma,\mathrm{TC}(A)(s,t)\}^{\mathbf{x}}}\ z\ \text{fresh}$$

$$\overline{\text{tc}}\ \frac{\mathbf{S},\{\Gamma,A(s,t),A(s,f(\mathbf{x}))\}^{\mathbf{x}},\{\Gamma,A(s,t),\overline{\mathrm{TC}}(A)(f(\mathbf{x}),t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\overline{\mathrm{TC}}(A)(s,t)\}^{\mathbf{x}}}\ f\ \text{fresh}$$

$$\begin{array}{lcl}\mathrm{TC}(A)(s,t) & \Longleftrightarrow & A(s,t)\vee\exists z(A(s,z)\wedge\mathrm{TC}(A)(z,t))\\ \overline{\mathrm{TC}}(A)(s,t) & \Longleftrightarrow & A(s,t)\wedge\forall z(A(s,z)\vee\overline{\mathrm{TC}}(A)(z,t))\end{array}$$

HTC preproofs

$$\text{init}\ \frac{}{\{\,\}^{\emptyset}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S},\mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\}^{\mathbf{x}},\{\overline{A}\}^{\emptyset}}\ A \text{ closed}$$

$$\cup\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}} \quad \mathbf{S},\{\Delta\}^{\mathbf{y}}}{\mathbf{S},\{\Gamma,\Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{array}{l}\mathrm{fv}(\Delta)\cap\mathbf{x}=\emptyset\\ \mathrm{fv}(\Gamma)\cap\mathbf{y}=\emptyset\end{array} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma,A,B\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S},\{\Gamma,A_i\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A_0\vee A_1\}^{\mathbf{x}}}\ i\in\{0,1\}$$

$$\forall\ \frac{\mathbf{S},\{\Gamma,A(f(\mathbf{x}))\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\forall z(A(z))\}^{\mathbf{x}}}\ f \text{ fresh} \qquad \exists\ \frac{\mathbf{S},\{\Gamma,A(y)\}^{\mathbf{x},y}}{\mathbf{S},\{\Gamma,\exists z(A(z))\}^{\mathbf{x}}}\ y \text{ fresh} \qquad \text{inst}\ \frac{\mathbf{S},\{\Gamma(t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma(y)\}^{\mathbf{x},y}}$$

$$\text{tc}\ \frac{\mathbf{S},\{\Gamma,A(s,t)\}^{\mathbf{x}},\{\Gamma,A(s,z),\mathrm{TC}(A)(z,t)\}^{\mathbf{x},z}}{\mathbf{S},\{\Gamma,\mathrm{TC}(A)(s,t)\}^{\mathbf{x}}}\ z \text{ fresh}$$

$$\overline{\text{tc}}\ \frac{\mathbf{S},\{\Gamma,A(s,t),A(s,f(\mathbf{x}))\}^{\mathbf{x}},\{\Gamma,A(s,t),\overline{\mathrm{TC}}(A)(f(\mathbf{x}),t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\overline{\mathrm{TC}}(A)(s,t)\}^{\mathbf{x}}}\ f \text{ fresh}$$

$$\begin{array}{lcl}\mathrm{TC}(A)(s,t) & \iff & A(s,t)\vee\exists z(A(s,z)\wedge\mathrm{TC}(A)(z,t))\\ \overline{\mathrm{TC}}(A)(s,t) & \iff & A(s,t)\wedge\forall z(A(s,z)\vee\overline{\mathrm{TC}}(A)(z,t))\end{array}$$

Trace $(F_i)_{i<\omega}$

HTC preproofs

$$\text{init}\ \frac{}{\{\,\}^{\emptyset}} \qquad \text{wk}\ \frac{\mathbf{S}}{\mathbf{S},\mathbf{S}'} \qquad \text{id}\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\}^{\mathbf{x}},\{\overline{A}\}^{\emptyset}}\ A \text{ closed}$$

$$\cup\ \frac{\mathbf{S},\{\Gamma\}^{\mathbf{x}} \quad \mathbf{S},\{\Delta\}^{\mathbf{y}}}{\mathbf{S},\{\Gamma,\Delta\}^{\mathbf{x},\mathbf{y}}}\ \begin{array}{l}\mathrm{fv}(\Delta)\cap\mathbf{x}=\emptyset\\ \mathrm{fv}(\Gamma)\cap\mathbf{y}=\emptyset\end{array} \qquad \wedge\ \frac{\mathbf{S}\{\Gamma,A,B\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A\wedge B\}^{\mathbf{x}}} \qquad \vee_i\ \frac{\mathbf{S},\{\Gamma,A_i\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,A_0\vee A_1\}^{\mathbf{x}}}\ i\in\{0,1\}$$

$$\forall\ \frac{\mathbf{S},\{\Gamma,A(f(\mathbf{x}))\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\forall z(A(z))\}^{\mathbf{x}}}\ f \text{ fresh} \qquad \exists\ \frac{\mathbf{S},\{\Gamma,A(y)\}^{\mathbf{x},y}}{\mathbf{S},\{\Gamma,\exists z(A(z))\}^{\mathbf{x}}}\ y \text{ fresh} \qquad \text{inst}\ \frac{\mathbf{S},\{\Gamma(t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma(y)\}^{\mathbf{x},y}}$$

$$\text{tc}\ \frac{\mathbf{S},\{\Gamma,A(s,t)\}^{\mathbf{x}},\{\Gamma,A(s,z),\mathrm{TC}(A)(z,t)\}^{\mathbf{x},z}}{\mathbf{S},\{\Gamma,\mathrm{TC}(A)(s,t)\}^{\mathbf{x}}}\ z \text{ fresh}$$

$$\overline{\text{tc}}\ \frac{\mathbf{S},\{\Gamma,A(s,t),A(s,f(\mathbf{x}))\}^{\mathbf{x}},\{\Gamma,A(s,t),\overline{\mathrm{TC}}(A)(f(\mathbf{x}),t)\}^{\mathbf{x}}}{\mathbf{S},\{\Gamma,\overline{\mathrm{TC}}(A)(s,t)\}^{\mathbf{x}}}\ f \text{ fresh}$$

$$\begin{array}{lcl}\mathrm{TC}(A)(s,t) & \iff & A(s,t)\vee\exists z(A(s,z)\wedge\mathrm{TC}(A)(z,t))\\ \overline{\mathrm{TC}}(A)(s,t) & \iff & A(s,t)\wedge\forall z(A(s,z)\vee\overline{\mathrm{TC}}(A)(z,t))\end{array}$$

Trace $(F_i)_{i<\omega}$ Hypertrace $(\{\Gamma_i\}^{\mathbf{x}_i})_{i<\omega}$

Defining progress in HTC

Trace $(F_i)_{i<\omega}$

Defining progress in HTC

Trace $(F_i)_{i<\omega}$ Hypertrace $(\{\Gamma_i\}^{\mathbf{x}_i})_{i<\omega}$

Defining progress in HTC

Trace $(F_i)_{i<\omega}$ Hypertrace $(\{\Gamma_i\}^{\mathbf{x}_i})_{i<\omega}$

Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $\overline{\mathrm{TC}}(A)(\cdot,\cdot)$

Defining progress in HTC

Trace $(F_i)_{i<\omega}$ Hypertrace $(\{\Gamma_i\}^{\mathbf{x}_i})_{i<\omega}$

Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $\overline{\mathrm{TC}}(A)(\cdot,\cdot)$

Progressing hypertrace: all infinite traces in it are progressing

Defining progress in HTC

Trace $(F_i)_{i<\omega}$ Hypertrace $(\{\Gamma_i\}^{\mathbf{x}_i})_{i<\omega}$

Progressing trace: The least formula occurring infinitely often principal in $(F_i)_{i<\omega}$ has the form $\overline{\mathrm{TC}}(A)(\cdot,\cdot)$

Progressing hypertrace: all infinite traces in it are progressing

Progress condition: Every infinite branch has a progressing hypertrace

Example

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\{\,\}^{\emptyset}$}
\LeftLabel{id}
\UnaryInfC{$\{a(c,d)\}^{\emptyset}, \{\bar{a}(c,d)\}^{\emptyset}$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\{\,\}^{\emptyset}$}
\LeftLabel{id}
\UnaryInfC{$\{a(c,e)\}^{\emptyset}, \{\bar{a}(c,e)\}^{\emptyset}$}
\AxiomC{$\vdots$}
\LeftLabel{TC}
\RightLabel{$\bullet$}
\UnaryInfC{$\{\overline{\mathrm{TC}}(a)(e,d)\}^{\emptyset}, \{\mathrm{TC}(\bar{a})(e,d)\}^{\emptyset}$}
\LeftLabel{$\cup$}
\BinaryInfC{$\{a(c,e)\}^{\emptyset}, \{\overline{\mathrm{TC}}(a)(e,d)\}^{\emptyset}, \{\bar{a}(c,e), \mathrm{TC}(\bar{a})(e,d)\}^{\emptyset}$}
\LeftLabel{$2\cup$}
\BinaryInfC{$\{a(c,d), a(c,e)\}^{\emptyset}, \{a(c,d), \overline{\mathrm{TC}}(a)(e,d)\}^{\emptyset}, \{\bar{a}(c,d)\}^{\emptyset}, \{\bar{a}(c,e), \mathrm{TC}(\bar{a})(e,d)\}^{\emptyset}$}
\LeftLabel{inst}
\UnaryInfC{$\{a(c,d), a(c,e)\}^{\emptyset}, \{a(c,d), \overline{\mathrm{TC}}(a)(e,d)\}^{\emptyset}, \{\bar{a}(c,d)\}^{\emptyset}, \{\bar{a}(c,x), \mathrm{TC}(\bar{a})(x,d)\}^{x}$}
\LeftLabel{$\overline{TC}$}
\UnaryInfC{$\{\overline{\mathrm{TC}}(a)(c,d)\}^{\emptyset}, \{\bar{a}(c,d)\}^{\emptyset}, \{\bar{a}(c,x), \mathrm{TC}(\bar{a})(x,d)\}^{x}$}
\LeftLabel{TC}
\RightLabel{$\bullet$}
\UnaryInfC{$\{\overline{\mathrm{TC}}(a)(c,d)\}^{\emptyset}, \{\mathrm{TC}(\bar{a})(c,d)\}^{\emptyset}$}
\LeftLabel{$\vee$}
\UnaryInfC{$\{\overline{\mathrm{TC}}(a)(c,d) \vee \mathrm{TC}(\bar{a})(c,d)\}^{\emptyset}$}
\end{prooftree}
$$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Main results: soundness

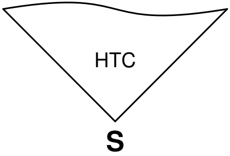
Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch

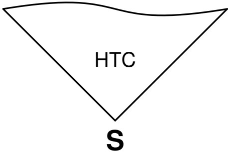


and **S** is not valid: $\mathcal{M}^{\times}, \rho_0 \not\models fm(\mathbf{S})$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch



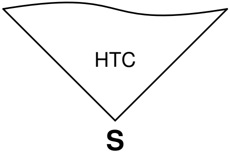
and **S** is not valid: $\mathcal{M}^{\times}, \rho_0 \not\models fm(\mathbf{S})$

- choose a branch $\mathcal{B}^{\times} = (\mathbf{S}_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times} \not\models fm(\mathbf{S}_i)$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch



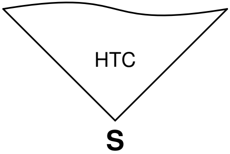
and **S** is not valid: $\mathcal{M}^{\times}, \rho_0 \not\models \mathit{fm}(\mathbf{S})$

- ▹ choose a branch $\mathcal{B}^{\times} = (\mathbf{S}_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times} \not\models \mathit{fm}(\mathbf{S}_i)$
- ▹ by definition, $\mathcal{B}^{\times}$ contains a progressing hypertrace $\mathcal{H}$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch



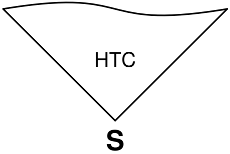
and **S** is not valid: $\mathcal{M}^{\times}, \rho_0 \not\models fm(\mathbf{S})$

- choose a branch $\mathcal{B}^{\times} = (\mathbf{S}_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times} \not\models fm(\mathbf{S}_i)$
- by definition, $\mathcal{B}^{\times}$ contains a progressing hypertrace $\mathcal{H}$
- choose a trace $\tau^{\times} = (F_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times}, \delta^{\mathcal{H}} \not\models F_i$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch



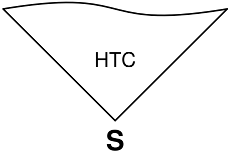
and **S** is not valid: $\mathcal{M}^\times, \rho_0 \not\models fm(\mathbf{S})$

- choose a branch $\mathcal{B}^\times = (\mathbf{S}_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^\times, \rho^\times \not\models fm(\mathbf{S}_i)$
- by definition, $\mathcal{B}^\times$ contains a progressing hypertrace $\mathcal{H}$
- choose a trace $\tau^\times = (F_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^\times, \rho^\times, \delta^{\mathcal{H}} \not\models F_i$
- by definition, $\tau^\times$ is progressing: for infinitely many i, $F_i = \overline{\mathsf{TC}}(A)(s_i, t)$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch



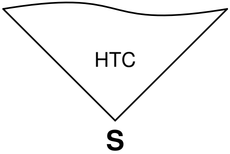
and **S** is not valid: $\mathcal{M}^{\times}, \rho_0 \not\models fm(\mathbf{S})$

- choose a branch $\mathcal{B}^{\times} = (\mathbf{S}_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times} \not\models fm(\mathbf{S}_i)$
- by definition, $\mathcal{B}^{\times}$ contains a progressing hypertrace $\mathcal{H}$
- choose a trace $\tau^{\times} = (F_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times}, \delta^{\mathcal{H}} \not\models F_i$
- by definition, $\tau^{\times}$ is progressing: for infinitely many i, $F_i = \overline{\mathrm{TC}}(A)(s_i, t)$
- contradiction: for infinitely many i, $\mathcal{M}^{\times}, \rho^{\times}, \delta^{\mathcal{H}} \models \mathrm{TC}(\bar{A})(s_i, t)$

Main results: soundness

Theorem (Soundness) Let **S** be a HTC hypersequent. If **S** has a cyclic proof in HTC, then *fm*(**S**) is valid in TCL.

Proof sketch



and **S** is not valid: $\mathcal{M}^{\times}, \rho_0 \not\models \mathit{fm}(\mathbf{S})$

- choose a branch $\mathcal{B}^{\times} = (\mathbf{S}_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times} \not\models \mathit{fm}(\mathbf{S}_i)$
- by definition, $\mathcal{B}^{\times}$ contains a progressing hypertrace $\mathcal{H}$
- choose a trace $\tau^{\times} = (F_i)_{i<\omega}$ s.t. for all i, $\mathcal{M}^{\times}, \rho^{\times}, \delta^{\mathcal{H}} \not\models F_i$
- by definition, $\tau^{\times}$ is progressing: for infinitely many i, $F_i = \overline{\mathsf{TC}}(A)(s_i, t)$
- contradiction: for infinitely many i, $\mathcal{M}^{\times}, \rho^{\times}, \delta^{\mathcal{H}} \models \mathsf{TC}(\bar{A})(s_i, t)$
- **S** is valid.

Can HTC simulate PDL^{+}?

Can HTC simulate PDL^{+}?

$$\begin{aligned} \mathrm{ST}(\langle a \rangle p)(x) &:= \exists y(a(x,y) \land p(y)) \\ \mathrm{ST}(\langle a_0 \cup a_1 \rangle p)(x) &:= \exists y((a_0(x,y) \lor a_1(x,y)) \land p(y)) \end{aligned}$$

Can HTC simulate PDL$^+$?

$$\mathrm{ST}(\langle a\rangle p)(x) \quad := \quad \exists y(a(x,y) \land p(y))$$

$$\mathrm{ST}(\langle a_0 \cup a_1\rangle p)(x) \quad := \quad \exists y((a_0(x,y) \lor a_1(x,y)) \land p(y))$$

$$\mathrm{HT}(\langle a\rangle p)(x) \quad := \quad \{a(x,y), p(y)\}^y$$

$$\mathrm{HT}(\langle a_0 \cup a_1\rangle p)(x) \quad := \quad \{a_0(x,y) \lor a_1(x,y), p(y)\}^y$$

Can HTC simulate PDL$^+$?

$$\begin{aligned}
\mathrm{ST}(\langle a\rangle p)(x) &:= \exists y(a(x,y) \wedge p(y)) \\
\mathrm{ST}(\langle a_0 \cup a_1\rangle p)(x) &:= \exists y((a_0(x,y) \vee a_1(x,y)) \wedge p(y))
\end{aligned}$$

$$\begin{aligned}
\mathrm{HT}(\langle a\rangle p)(x) &:= \{a(x,y), p(y)\}^y \\
\mathrm{HT}(\langle a_0 \cup a_1\rangle p)(x) &:= \{a_0(x,y) \vee a_1(x,y), p(y)\}^y
\end{aligned}$$

$$\langle\cup\rangle_i \frac{\langle a_i\rangle p}{\langle a_0 \cup a_1\rangle p}\, i \in \{0,1\} \qquad \rightsquigarrow$$

Can HTC simulate PDL$^+$?

$$\begin{aligned}
\mathrm{ST}(\langle a\rangle p)(x) &:= \exists y(a(x,y) \wedge p(y)) \\
\mathrm{ST}(\langle a_0 \cup a_1\rangle p)(x) &:= \exists y((a_0(x,y) \vee a_1(x,y)) \wedge p(y))
\end{aligned}$$

$$\begin{aligned}
\mathrm{HT}(\langle a\rangle p)(x) &:= \{a(x,y), p(y)\}^y \\
\mathrm{HT}(\langle a_0 \cup a_1\rangle p)(x) &:= \{a_0(x,y) \vee a_1(x,y), p(y)\}^y
\end{aligned}$$

$$\langle\cup\rangle_i \, \frac{\langle a_i\rangle p}{\langle a_0 \cup a_1\rangle p} \; i \in \{0,1\} \qquad \rightsquigarrow \qquad \vee_i \, \frac{\dfrac{\mathrm{HT}(\langle a_i\rangle p)(x)}{\{a_i(x,y), p(y)\}^y}}{\dfrac{\{a_0(x,y) \vee a_1(x,y), p(y)\}^y}{\mathrm{HT}(\langle a_0 \cup a_1\rangle p)(x)}}$$

Can HTC simulate PDL$^+$?

$$\begin{aligned}
\mathrm{ST}(\langle a \rangle p)(x) &:= \exists y (a(x,y) \wedge p(y)) \\
\mathrm{ST}(\langle a_0 \cup a_1 \rangle p)(x) &:= \exists y ((a_0(x,y) \vee a_1(x,y)) \wedge p(y))
\end{aligned}$$

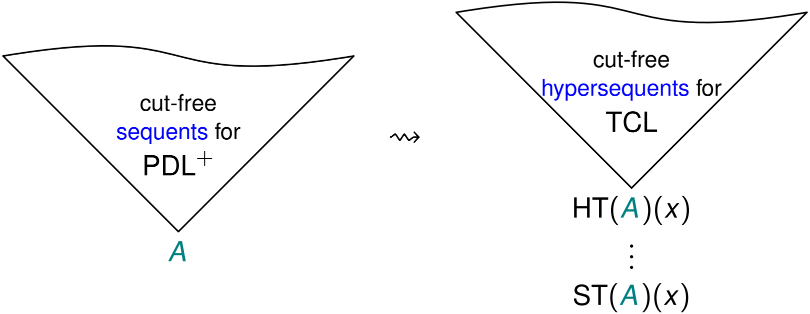
$$\begin{aligned}
\mathrm{HT}(\langle a \rangle p)(x) &:= \{a(x,y), p(y)\}^y \\
\mathrm{HT}(\langle a_0 \cup a_1 \rangle p)(x) &:= \{a_0(x,y) \vee a_1(x,y), p(y)\}^y
\end{aligned}$$

$$\langle \cup \rangle_i \, \frac{\langle a_i \rangle p}{\langle a_0 \cup a_1 \rangle p} \; i \in \{0,1\} \qquad \rightsquigarrow \qquad \vee_i \, \frac{\dfrac{\mathrm{HT}(\langle a_i \rangle p)(x)}{\{a_i(x,y), p(y)\}^y}}{\dfrac{\{a_0(x,y) \vee a_1(x,y), p(y)\}^y}{\mathrm{HT}(\langle a_0 \cup a_1 \rangle p)(x)}}$$

$$\vee_i \, \frac{\mathbf{S}, \{\Gamma, A_i\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma, A_0 \vee A_1\}^{\mathbf{x}}} \; i \in \{0,1\}$$

Main results: simulation and completeness

Theorem (Simulation of PDL$^+$)



cut-free
sequents for
PDL$^+$

A

$\rightsquigarrow$

cut-free
hypersequents for
TCL

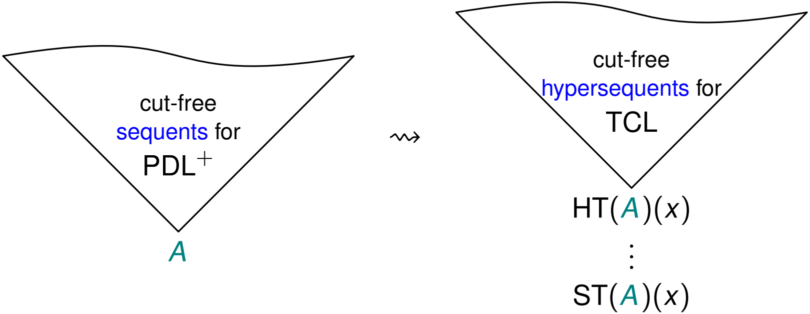
$\mathsf{HT}(A)(x)$

$\vdots$

$\mathsf{ST}(A)(x)$

Main results: simulation and completeness

Theorem (Simulation of PDL$^+$)



cut-free sequents for PDL$^+$

A

$\rightsquigarrow$

cut-free hypersequents for TCL

$\mathrm{HT}(A)(x)$

$\vdots$

$\mathrm{ST}(A)(x)$

Corollary (Completeness) If $\mathrm{ST}(A)(x)$ is valid in TCL, then it has a cyclic proof in HTC.

Conclusions

Conclusions

This talk

Conclusions

This talk

- HTC, cyclic hypersequents for TCL

Conclusions

This talk

- HTC, cyclic hypersequents for TCL
- HTC simulates PDL^{+}

Conclusions

This talk

- HTC, cyclic hypersequents for TCL
- HTC simulates PDL^{+}
- Soundness of HTC

Conclusions

This talk

- HTC, cyclic hypersequents for TCL
- HTC simulates PDL^{+}
- Soundness of HTC
- (Modular extension of all the above to TCL$^{=}$ and PDL)

Conclusions

This talk

- HTC, cyclic hypersequents for TCL
- HTC simulates PDL^{+}
- Soundness of HTC
- (Modular extension of all the above to $\text{TCL}^{=}$ and PDL)

Future work

Conclusions

This talk

- HTC, cyclic hypersequents for TCL
- HTC simulates PDL^{+}
- Soundness of HTC
- (Modular extension of all the above to $\mathrm{TCL}^{=}$ and PDL)

Future work

- Study the complexity of the simulation

Conclusions

This talk

- HTC, cyclic hypersequents for TCL
- HTC simulates PDL$^+$
- Soundness of HTC
- (Modular extension of all the above to TCL$^=$ and PDL)

Future work

- Study the complexity of the simulation
- Study the proof-theoretical properties of cyclic hypersequents

Thank you!

Questions?

Bonus: adding equality

☞ Adding tests to the language of PDL$^+$

$$\alpha, \beta := a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^+ \mid A?$$

$$\alpha^* := (\top? \cup \alpha)^+$$

$$\langle ? \rangle \frac{\Gamma, A \quad \Gamma, B}{\Gamma, \langle A? \rangle B} \qquad [?] \frac{\Gamma, \bar{A}, B}{\Gamma, [A?]B}$$

☞ Add equality to the language of TCL

$$\mathrm{RTC}(A)(s,t) := \mathrm{TC}((x = y \vee A))(s,t)$$

$$= \frac{\mathbf{S}, \{\Gamma\}^{\mathbf{x}}}{\mathbf{S}, \{t = t, \Gamma\}^{\mathbf{x}}} \qquad \neq \frac{\mathbf{S}, \{\Gamma(s), \Delta(s)\}^{\mathbf{x}}}{\mathbf{S}, \{\Gamma(s), s \neq t\}^{\mathbf{x}}, \{\Delta(t)\}^{\mathbf{x}}}$$

Theorem (Soundness) If **S** has a cyclic proof in HTC$_=$, then *fm*(**S**) is valid in RTCL.

Theorem (Completeness) Let A be a PDL formula. If A is valid in PDL, then ST$(A)(x)$ has a cyclic proof in HTC$_=$.