Intuitionistic S4 is decidable!

Marianna Girlando
Universtiy of Amsterdam

Roman Kuznets
TU Wien

Sonia Marin
University of Birmingham

Marianela Morales
Inria Saclay

Lutz Straßburger
Inria Saclay

LLAMA Seminar
Amsterdam, 25 October 2023

In a nutshell

Is IS4 decidable? [Simpson, 1994]

Yes!

https://arxiv.org/abs/2304.12094

Intuitionistic modal logics

[Fischer Servi, 1984], [Plotkin & Stirling, 1986], [Simpson, 1994]

$\mathcal{A}$ set of propositional variables, $p \in \mathcal{A}$

$\mathcal{L} ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A \qquad \neg A \equiv A \supset \bot$

nec : if A is provable, so is $\Box A$

k_1 : $\Box(A \supset B) \supset (\Box A \supset \Box B)$

k_2 : $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

k_3 : $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$

k_4 : $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

k_5 : $\Diamond\bot \supset \bot$

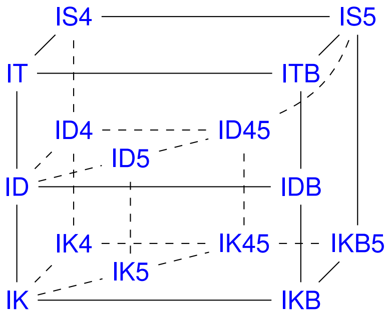
d : $\Box A \supset \Diamond A$

t : $\Box A \supset A \quad \wedge \quad A \supset \Diamond A$

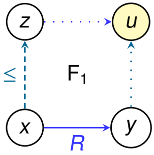
b : $A \supset \Box\Diamond A \quad \wedge \quad \Diamond\Box A \supset A$

4 : $\Box A \supset \Box\Box A \quad \wedge \quad \Diamond\Diamond A \supset \Diamond A$

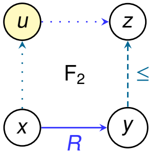
5 : $\Diamond A \supset \Box\Diamond A \quad \wedge \quad \Diamond\Box A \supset \Box A$



Bi-relational models for IS4 [Fischer Servi, 1984]



$$\mathcal{M} = \langle W, R, \leq, V \rangle$$



- W non-empty set
- R reflexive and transitive relation (S4 models)
- $\leq$ reflexive and transitive relation (IPL models)
- $V : W \longrightarrow 2^{\mathcal{A}}$ s.t. if $x \leq y$ then $V(x) \subseteq V(y)$ (IPL models)

$\mathcal{M}, w \nVdash \bot$;
$\mathcal{M}, w \Vdash p$ iff $p \in V(w)$;
$\mathcal{M}, w \Vdash A \wedge B$ iff $\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \vee B$ iff $\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \supset B$ iff for all w' with $w \leq w'$, if $\mathcal{M}, w' \Vdash A$, then $\mathcal{M}, w' \Vdash B$;
$\mathcal{M}, w \Vdash \Diamond A$ iff there exists u such that wRu and $\mathcal{M}, u \Vdash A$;
$\mathcal{M}, w \Vdash \Box A$ iff for all w', u with $w \leq w'$ and $w'Ru$, we have $\mathcal{M}, u \Vdash A$.

Monotonicity: if $\mathcal{M}, x \Vdash A$ and $x \leq y$, then $\mathcal{M}, x \Vdash A$

A is valid iff for all $\mathcal{M}$ and W it holds that $\mathcal{M}, w \Vdash A$

Fully labelled sequent calculus [Marin, Morales, Straßburger, 2021]

Enriching the language: given countably many variables (labels), define:

- Relational atoms xRy, $x \leq y$
- Labelled formulas $x{:}A$

A labelled sequent $\mathcal{G}$ is a triple $\mathcal{R}, \Gamma \Longrightarrow \Delta$ with:

- $\mathcal{R}$ multiset of relational atoms, and
- Γ, Δ multiset of labelled formulas labelled formulas

Shorthands: for a sequent $\mathcal{G}$, we write

- $x \leq_{\mathcal{G}} y$ iff $x \leq y$ occurs in $\mathcal{R}$;
- $x\ R_{\mathcal{G}}\ y$ iff xRy occurs in $\mathcal{R}$;
- $\mathcal{G}, x{:}A^{\bullet}$ iff $x{:}A$ occurs in Γ;
- $\mathcal{G}, x{:}A^{\circ}$ iff $x{:}A$ occurs in Δ.

Labels x and y are equivalent, written $x \sim y$, iff, for all formulas A,

- $\mathcal{G}, x{:}A^{\bullet} \Longleftrightarrow \mathcal{H}, y{:}A^{\bullet}$, and
- $\mathcal{G}, x{:}A^{\circ} \Longleftrightarrow \mathcal{H}, y{:}A^{\circ}$.

Rules of labIS4$_{\le}$

$$\text{id}\ \frac{}{\mathcal{R}, x{\le}y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a} \qquad \bot^{\bullet}\ \frac{}{\mathcal{R}, \Gamma, x{:}\bot \Longrightarrow \Delta}$$

$$\wedge^{\bullet}\ \frac{\mathcal{R}, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}A \wedge B \Longrightarrow \Delta} \qquad \wedge^{\circ}\ \frac{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \wedge B}$$

$$\vee^{\bullet}\ \frac{\mathcal{R}, \Gamma, x{:}A \Longrightarrow \Delta \quad \mathcal{R}, \Gamma, x{:}B \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}A \vee B \Longrightarrow \Delta} \qquad \vee^{\circ}\ \frac{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \vee B}$$

$$\supset^{\bullet}\ \frac{\mathcal{R}, x{\le}y, x{:}A \supset B, \Gamma \Longrightarrow \Delta, y{:}A \quad \mathcal{R}, x{\le}y, \Gamma, y{:}B \Longrightarrow \Delta}{\mathcal{R}, x{\le}y, \Gamma, x{:}A \supset B \Longrightarrow \Delta} \qquad \supset^{\circ}\ \frac{\mathcal{R}, x{\le}z, \Gamma, z{:}A \Longrightarrow \Delta, z{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B}\ z \text{ fresh}$$

$$\Box^{\bullet}\ \frac{\mathcal{R}, x{\le}y, yRz, \Gamma, x{:}\Box A, z{:}A \Longrightarrow \Delta}{\mathcal{R}, x{\le}y, yRz, \Gamma, x{:}\Box A \Longrightarrow \Delta} \qquad \Box^{\circ}\ \frac{\mathcal{R}, x{\le}u, uRz, \Gamma \Longrightarrow \Delta, z{:}A}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A}\ u, z \text{ fresh}$$

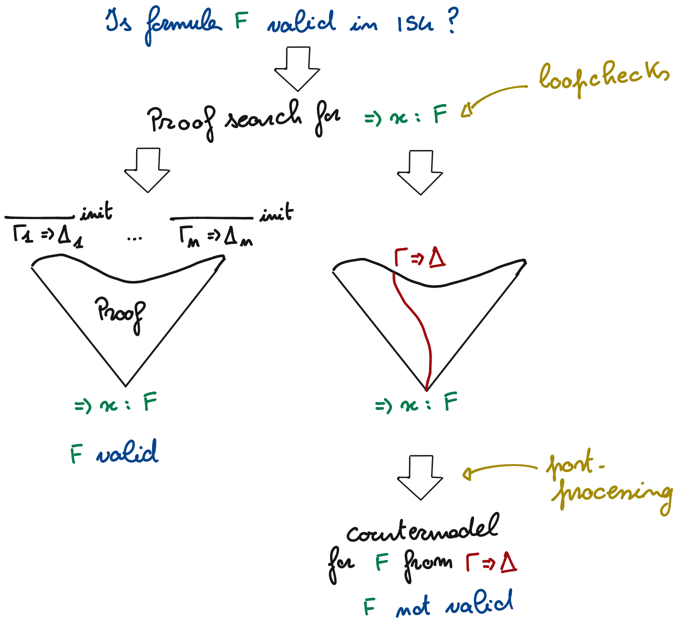
$$\Diamond^{\bullet}\ \frac{\mathcal{R}, xRy, \Gamma, y{:}A \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta}\ y \text{ fresh} \qquad \Diamond^{\circ}\ \frac{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, y{:}A}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A}$$

$$\text{F}_1\ \frac{\mathcal{R}, xRy, y{\le}z, x{\le}u, uRz, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, y{\le}z, \Gamma \Longrightarrow \Delta}\ u \text{ fresh} \qquad \text{F}_2\ \frac{\mathcal{R}, xRy, x{\le}z, y{\le}u, zRu, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, x{\le}z, \Gamma \Longrightarrow \Delta}\ u \text{ fresh}$$

$$\le\text{rf}\ \frac{\mathcal{R}, x{\le}x, \Gamma \Longrightarrow \Delta}{\mathcal{R}, \Gamma \Longrightarrow \Delta} \qquad \le\text{tr}\ \frac{\mathcal{R}, x{\le}y, y{\le}z, x{\le}z, \Gamma \Longrightarrow \Delta}{\mathcal{R}, x{\le}y, y{\le}z, \Gamma \Longrightarrow \Delta}$$

$$\mathcal{R}\text{rf}\ \frac{\mathcal{R}, xRx, \Gamma \Longrightarrow \Delta}{\mathcal{R}, \Gamma \Longrightarrow \Delta} \qquad \mathcal{R}\text{tr}\ \frac{\mathcal{R}, xRy, yRz, xRz, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, yRz, \Gamma \Longrightarrow \Delta}$$

A proof theoretic approach to decidability



Loopcheck for transitive R [Ladner, 1977]

Is $\Box\Diamond a \supset \bot$ valid in IS4?

$$\vdots$$

$$\Diamond^\bullet \ \frac{}{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}$$

$$\Box^\bullet \ \frac{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}$$

$$\mathcal{R}\mathrm{tr} \ \frac{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}$$

$$\Diamond^\bullet \ \frac{2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}$$

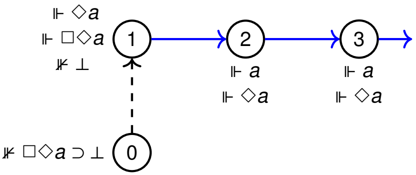
$$\Box^\bullet \ \frac{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}$$

$$\Diamond^\bullet \ \frac{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}{1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}$$

$$\Box^\bullet \ \frac{1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}{1R1, 0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}$$

$$\mathcal{R}\mathrm{rf} \ \frac{1R1, 0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}$$

$$\supset^\circ \ \frac{0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{\Longrightarrow 0{:}\Box\Diamond a \supset \bot}$$



Loopcheck for transitive R [Ladner, 1977]

Is $\Box\Diamond a \supset \bot$ valid in IS4?

$$\vdots$$

$$\Diamond^{\bullet}\ \frac{\qquad\vdots\qquad}{1R3, 2R3, 1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}$$

$$\Box^{\bullet}\ \frac{1R3, 2R3, 1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}{1R3, 2R3, 1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}$$

$$\mathcal{R}\mathrm{tr}\ \frac{1R3, 2R3, 1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{2R3, 1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}$$

$$\Diamond^{\bullet}\ \frac{2R3, 1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}$$

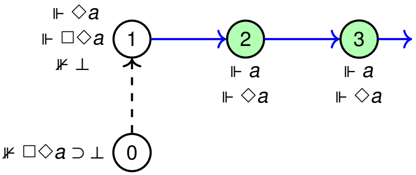
$$\Box^{\bullet}\ \frac{1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}$$

$$\Diamond^{\bullet}\ \frac{1R2, 1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}{1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}$$

$$\Box^{\bullet}\ \frac{1R1, 0{\leq}1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}{1R1, 0{\leq}1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}$$

$$\mathcal{R}\mathrm{rf}\ \frac{1R1, 0{\leq}1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{0{\leq}1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}$$

$$\supset^{\circ}\ \frac{0{\leq}1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{\Longrightarrow 0{:}\Box\Diamond a \supset \bot}$$



Loopcheck for transitive R [Ladner, 1977]

Is $\Box\Diamond a \supset \bot$ valid in IS4?

$$\frac{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}\ \Box^\bullet$$

$$\frac{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}\ \mathcal{R}\mathrm{tr}$$

$$\frac{2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}\ \Diamond^\bullet$$

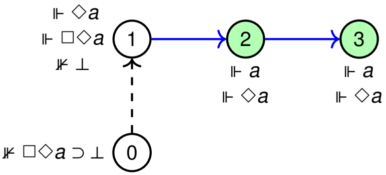
$$\frac{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}\ \Box^\bullet$$

$$\frac{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}{1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}\ \Diamond^\bullet$$

$$\frac{1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}{1R1, 0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}\ \Box^\bullet$$

$$\frac{1R1, 0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}\ \mathcal{R}\mathrm{rf}$$

$$\frac{0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{\Longrightarrow 0{:}\Box\Diamond a \supset \bot}\ \supset^\circ$$



Loopcheck for transitive R [Ladner, 1977]

Is $\Box\Diamond a \supset \bot$ valid in IS4?

$$\dfrac{}{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}$$

$$\Box^\bullet \; \dfrac{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a, 3{:}\Diamond a \Longrightarrow 1{:}\bot}{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}$$

$$\mathcal{R}\mathrm{tr} \; \dfrac{1R3, 2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}$$

$$\Diamond^\bullet \; \dfrac{2R3, 1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a, 3{:}a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}$$

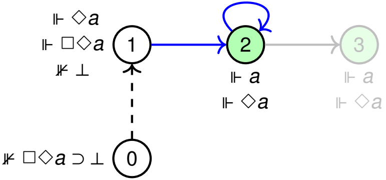
$$\Box^\bullet \; \dfrac{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a, 2{:}\Diamond a \Longrightarrow 1{:}\bot}{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}$$

$$\Diamond^\bullet \; \dfrac{1R2, 1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a, 2{:}a \Longrightarrow 1{:}\bot}{1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}$$

$$\Box^\bullet \; \dfrac{1R1, 0\leq 1, 1{:}\Box\Diamond a, 1{:}\Diamond a \Longrightarrow 1{:}\bot}{1R1, 0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}$$

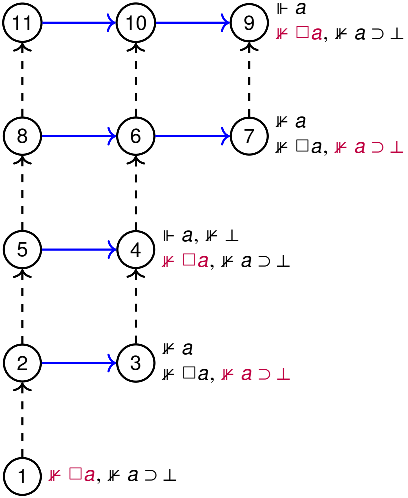
$$\mathcal{R}\mathrm{rf} \; \dfrac{1R1, 0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}$$

$$\supset^\circ \; \dfrac{0\leq 1, 1{:}\Box\Diamond a \Longrightarrow 1{:}\bot}{\Longrightarrow 0{:}\Box\Diamond a \supset \bot}$$



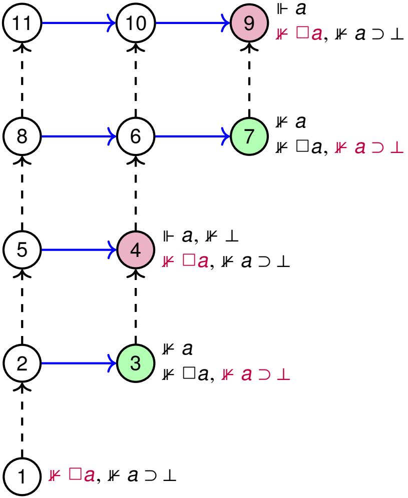
Loopcheck for transitive R and transitive $\leq$

Is $\Box\Big((\Box a \supset \bot) \wedge \big((a \supset \bot) \supset \bot\big)\Big) \supset \bot$ valid in IS4?



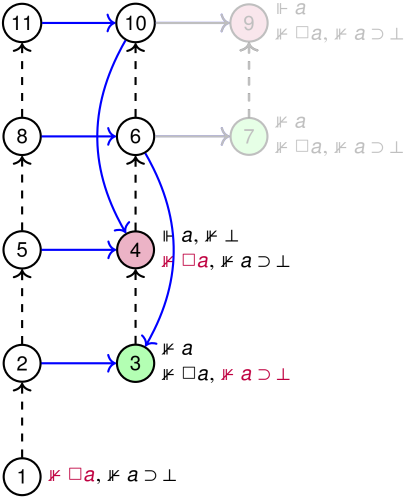
Loopcheck for transitive R and transitive $\leq$

Is $\Box\Big((\Box a \supset \bot) \wedge \big((a \supset \bot) \supset \bot\big)\Big) \supset \bot$ valid in IS4?



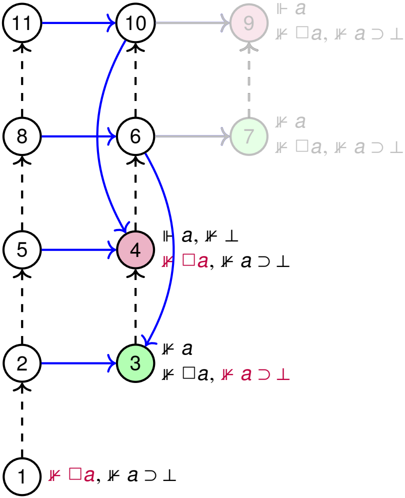
Loopcheck for transitive R and transitive $\leq$

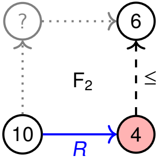
Is $\Box\Big((\Box a \supset \bot) \wedge \big((a \supset \bot) \supset \bot\big)\Big) \supset \bot$ valid in IS4?



Loopcheck for transitive R and transitive $\leq$

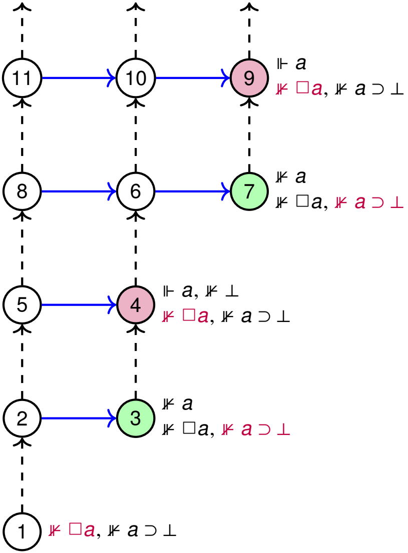
Is $\Box\Big((\Box a \supset \bot) \wedge \big((a \supset \bot) \supset \bot\big)\Big) \supset \bot$ valid in IS4?



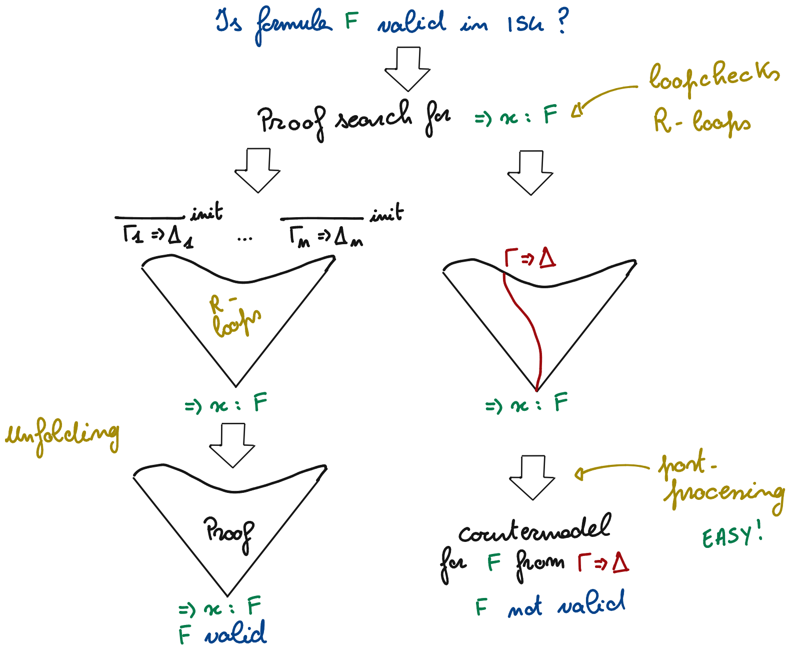


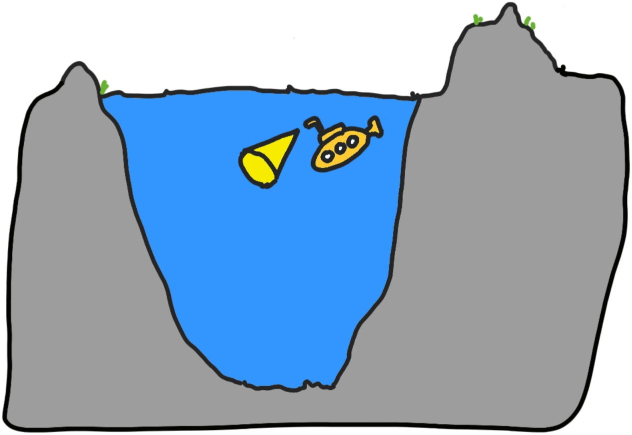
Loopcheck for transitive R and transitive $\leq$

Is $\Box\Big((\Box a \supset \bot) \wedge \big((a \supset \bot) \supset \bot\big)\Big) \supset \bot$ valid in IS4?



Our approach to decidability

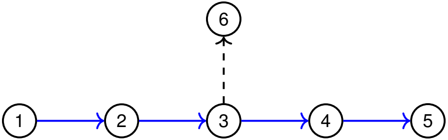




Main ingredients

Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

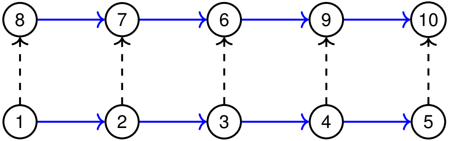


- First saturate an existing layer (rules $\wedge^\bullet$, $\wedge^\circ$, $\vee^\bullet$, $\vee^\circ$, $\supset^\bullet$, $\Box^\bullet$, $\Diamond^\circ$, $\Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ$, $\Box^\circ$).

Main ingredients

Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

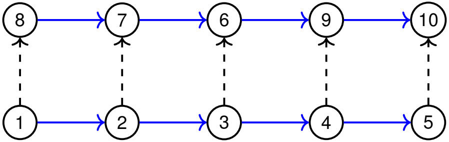


- First saturate an existing layer (rules $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \supset^\bullet, \Box^\bullet, \Diamond^\circ, \Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ, \Box^\circ$).

Main ingredients

Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

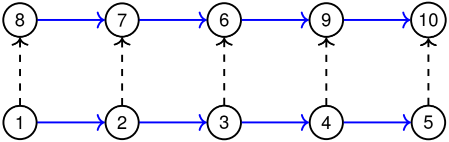


- First saturate an existing layer (rules $\wedge^\bullet$, $\wedge^\circ$, $\vee^\bullet$, $\vee^\circ$, $\supset^\bullet$, $\Box^\bullet$, $\Diamond^\circ$, $\Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ$, $\Box^\circ$).

Main ingredients

Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

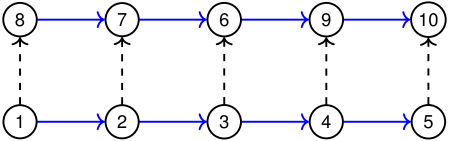


- First saturate an existing layer (rules $\wedge^\bullet$, $\wedge^\circ$, $\vee^\bullet$, $\vee^\circ$, $\supset^\bullet$, $\Box^\bullet$, $\Diamond^\circ$, $\Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ$, $\Box^\circ$).

Main ingredients

Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

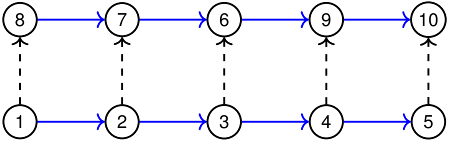


- First saturate an existing layer (rules $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \supset^\bullet, \Box^\bullet, \Diamond^\circ, \Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ, \Box^\circ$).

Main ingredients

Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

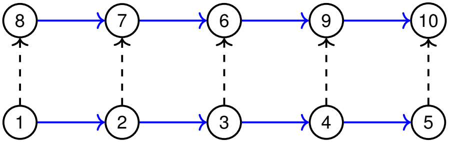


- First saturate an existing layer (rules $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \supset^\bullet, \Box^\bullet, \Diamond^\circ, \Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ, \Box^\circ$).
  A layer can be lifted only if it is not simulated by a previous layer.

Main ingredients

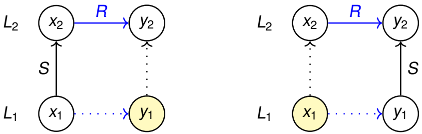
Proof search is structured into layers

Let $\hat{R}_{\mathcal{G}}$ be the transitive and reflexive closure of $R_{\mathcal{G}} \cup R_{\mathcal{G}}^{-1}$.
$\hat{R}_{\mathcal{G}}$ is an equivalence relation; a layer is an equivalence class of $\hat{R}_{\mathcal{G}}$.

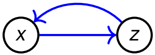


- First saturate an existing layer (rules $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \supset^\bullet, \Box^\bullet, \Diamond^\circ, \Diamond^\bullet$);
- Then generate a new layer by lifting an existing layer (rules $\supset^\circ, \Box^\circ$).
  A layer can be lifted only if it is not simulated by a previous layer.

Layer L_1 simulates layer L_2 iff there is a relation $S \subseteq (L_1 \times L_2) \cap \sim$ s.t.:



We introduce clusters (equivalence classes of $R_{\mathcal{G}} \cap R_{\mathcal{G}}^{-1}$).



The Algorithm

Is formula F valid in IS4?

0. Let $\mathcal{G}_0(F)$ be the sequent $r \leq r \Longrightarrow r{:}F$ and let $\mathfrak{S}'_0 = \{\mathcal{G}_0(F)\}$.

1. For the set $\mathfrak{S}'_i$, calculate a saturation $\mathfrak{S}_i$.

   Apply $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \supset^\bullet, \Box^\bullet, \Diamond^\circ$ and $\Diamond^\bullet$ "as much as possible"; introduce R-loops

2. If all sequents in $\mathfrak{S}_i$ are axiomatic, then terminate. $\rightsquigarrow$ F is valid

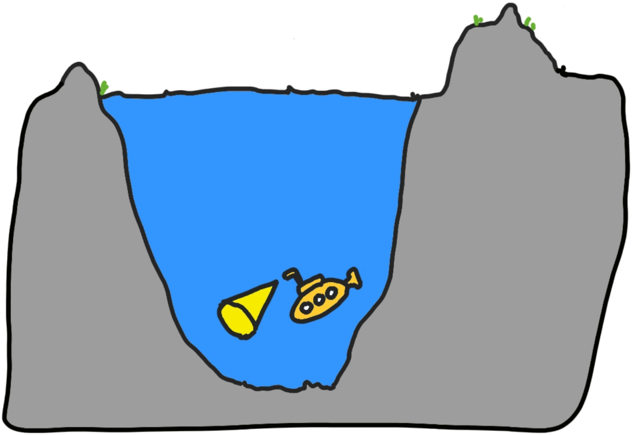
3. Otherwise pick a non-axiomatic sequent $\mathcal{G}_i \in \mathfrak{S}_i$.

   3.1 Pick an allowed formula $x{:}A^\circ$ in $\mathcal{G}_i$, compute the lifting $\mathcal{G}_i{\uparrow}^{x:A^\circ}$, and let $\mathfrak{S}'_{i+1} = (\mathfrak{S}_i \setminus \{\mathcal{G}_i\}) \cup \{\mathcal{G}_i + \mathcal{G}_i{\uparrow}^{x:A^\circ}\}$. Go to Step 1.

   Apply one of $\supset^\circ, \Box^\circ$

   to formulas occurring in layers which are not simulated

   3.2 Otherwise, if there are no more allowed formulas in $\mathcal{G}_i$, then terminate. $\rightsquigarrow$ F is not valid



Saturation

1. For the set $\mathfrak{S}'_i$, calculate its saturation $\mathfrak{S}_i$.

   a. For the set $\mathfrak{S}'_i$, calculate its semi-saturation $\mathfrak{S}_i$.
   Apply “as much as possible” rules $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \supset^\bullet, \Box^\bullet, \Diamond^\circ$.

   b. If there are $\Diamond^\bullet$-formulas in some non-axiomatic sequent $\mathcal{G}_i \in \mathfrak{S}_i$ to which the $\Diamond^\bullet$ rule can be applied, then pick one such formula, $y{:}\Diamond A^\bullet$.
   Compute the 4-saturation $\mathcal{G}_i^{\Diamond^\bullet}$ of $\mathcal{G}_i$ and let $\mathfrak{S}'_{i+1} = (\mathfrak{S}_i \setminus \{\mathcal{G}_i\}) \cup \{\mathcal{G}_i^{\Diamond^\bullet}\}$.
   Go to Step a.

   c. If there are no such formulas, calculate the loop saturation of $\mathfrak{S}'_{i+1}$.

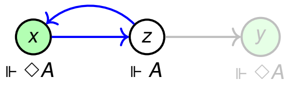
4-saturation

Let $\mathcal{G}$ be a sequent containing a formula $y{:}\Diamond A^{\bullet}$ to which the $\Diamond^{\bullet}$ rule can be applied (and such that, for all the labels u with $u\, R_{\mathcal{G}}\, x$ and $u \neq x$, no rule can be applied to $u{:}B^{\bullet}$, $u{:}B^{\circ}$, except for $\supset^{\circ}$, $\Box^{\circ}$).

The $\Diamond$-saturation $\mathcal{G}^{\Diamond^{\bullet}}$ of $\mathcal{G}$ is obtained by either:

- If there is a label $x \neq y$ with $x\, R_{\mathcal{G}}\, y$ and $x \sim y$ (and every label w with $x\, R_{\mathcal{G}}\, w$ has no past in $\mathcal{G}$), substitute each occurrence of y with x and close under $\mathcal{R}$tr;

$$\mathcal{R}\text{tr}\;\dfrac{\dfrac{xRx, xRz, \mathcal{R}'', x{:}\Diamond A, z{:}A, x{:}\Diamond A, \Gamma \Longrightarrow \Delta}{xRx, xRz, \mathcal{R}', x{:}\Diamond A, z{:}A, x{:}\Diamond A, \Gamma \Longrightarrow \Delta}}{xRy, xRz, \mathcal{R}, x{:}\Diamond A, z{:}A, y{:}\Diamond A, \Gamma \Longrightarrow \Delta}$$

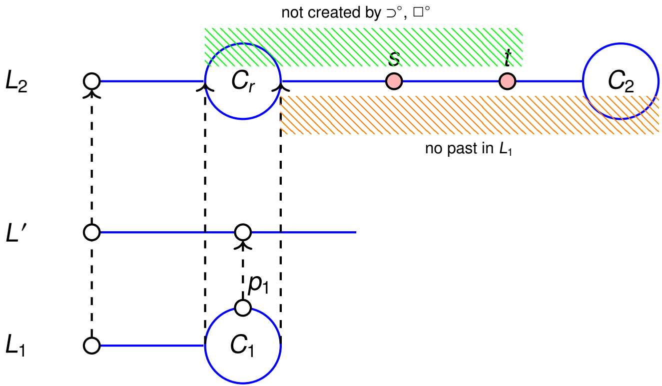


- Otherwise, apply rule $\Diamond^{\bullet}$ and close under structural rules.

$$\Diamond^{\bullet}\;\dfrac{\text{struc}\;\dfrac{z{\leq}z, yRz, \mathcal{R}, z{:}A, y{:}\Diamond A, \Gamma' \Longrightarrow \Delta}{yRz, \mathcal{R}, z{:}A, y{:}\Diamond A, \Gamma \Longrightarrow \Delta}}{\mathcal{R}, y{:}\Diamond A, \Gamma \Longrightarrow \Delta}$$

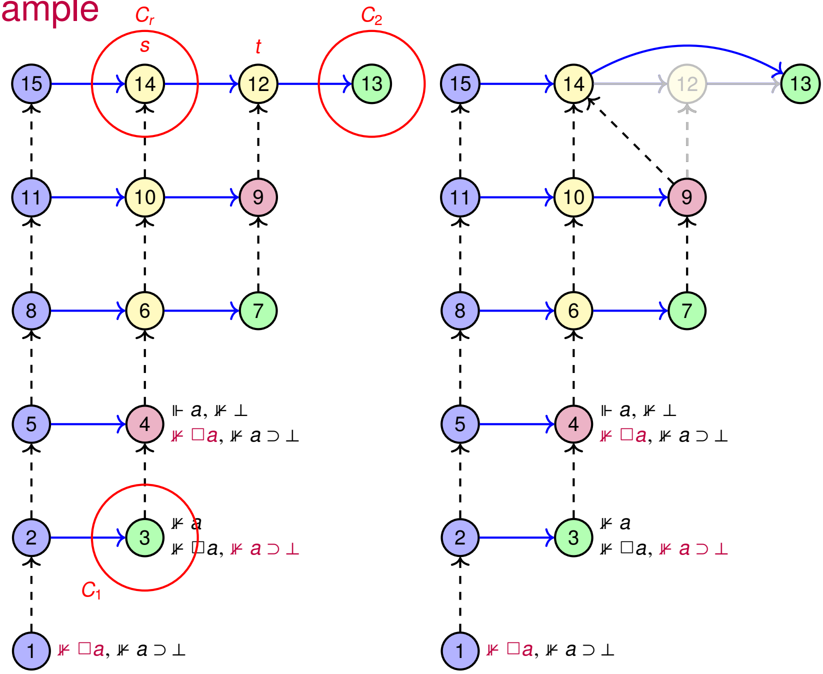
Loop saturation

Let $\mathcal{G}$ be the following sequent, where $C_1 \sim C_2$ and $s \sim t$:



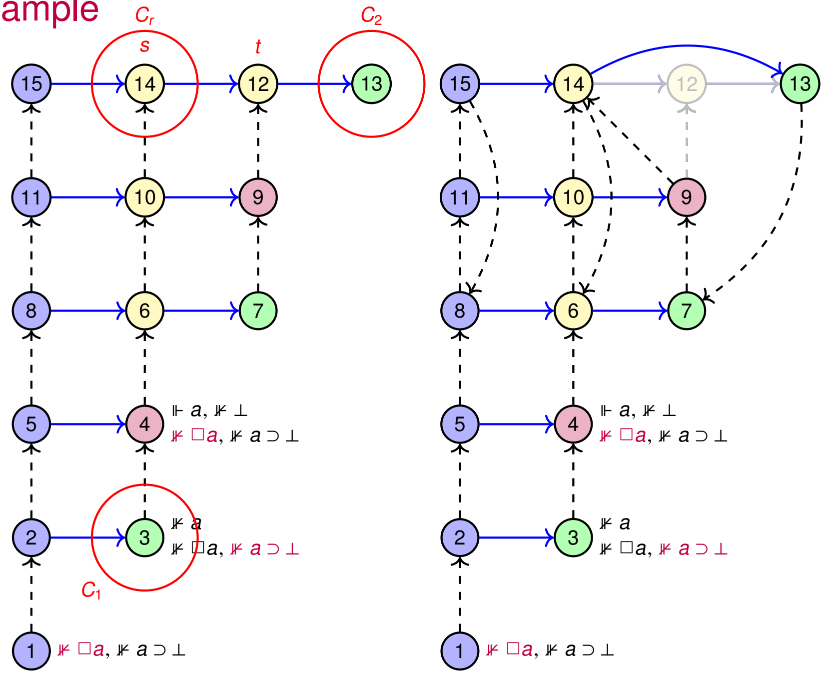
The loop saturation $\mathcal{G}^{\circlearrowleft}$ of $\mathcal{G}$ is obtained by substituting each occurrence of t with s and closing under $\mathcal{R}$tr.

Example



C_r
s
t
C_2
15
14
12
13
11
10
9
8
6
7
5
4
$\Vdash a, \nVdash \bot$
$\nVdash \Box a, \nVdash a \supset \bot$
2
3
$\nVdash a$
$\nVdash \Box a, \nVdash a \supset \bot$
C_1
1
$\nVdash \Box a, \nVdash a \supset \bot$

Example



C_r
C_2
s
t
15
14
12
13
11
10
9
8
6
7
5
4
$\Vdash a, \nVdash \bot$
$\nVdash \Box a, \nVdash a \supset \bot$
2
3
$\nVdash a$
$\nVdash \Box a, \nVdash a \supset \bot$
C_1
1
$\nVdash \Box a, \nVdash a \supset \bot$

To prove

Termination The Algorithm is terminating.

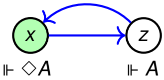
Completeness If the Algorithm terminates in Step 4, then formula F is not a theorem of IS4.
This is easy!

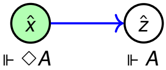
Soundness If the Algorithm terminates in Step 2, then the formula F is a theorem of IS4.
Here we need to use unfoldings.

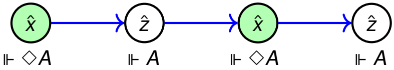
Unfoldings

Goal: proof with only of proper sequents (only singleton clusters)

n-Unfolding of $\mathcal{G}$: proper sequent $\hat{\mathcal{G}}$ together with $\mathrm{U} \subseteq \ell(\mathcal{G}) \times \ell(\hat{\mathcal{G}})$, obeying some conditions, unwinding clusters in $\mathcal{G}$ n times.

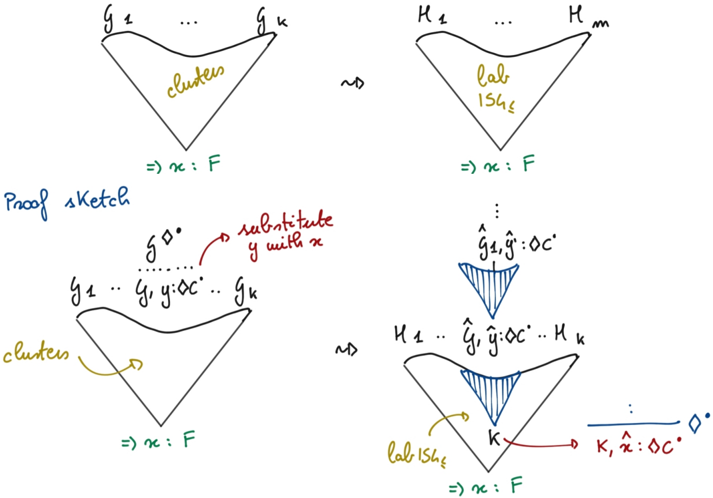


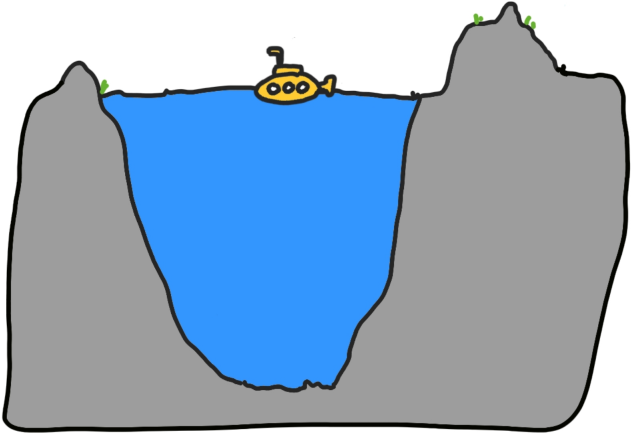




Unfolding Theorem

Given a set of sequents $\mathcal{G}_1, \ldots, \mathcal{G}_k$ produced by the algorithm, for every $n \geq 1$ there is a labIS4$_\leq$ derivation of premisses $\mathcal{H}_1, \ldots, \mathcal{H}_m$, such that each $\mathcal{H}_j$, for $j \in \{1..m\}$, is an n-unfolding of some $\mathcal{G}_i$, for $i \in \{1..k\}$.





Thank you!