Quantum Monadic Algebras

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Overview

This is a first step in a larger project.

Aim: look at modern aspects of operator algebras from the historically useful perspective of lattices, geometry and logic.

vN: modern Hilbert space, operators

vN: foundation of QM in Hilbert space

1930 ———

vN, Stone: modern spectral theorem

Stone: representations of Boolean algebras

B, Menger: finite-dimensional lattice-theoretic projective geometry

B, vN: logic of QM

vN: continuous geometry, rep's of complemented modular lattices

M, vN: Rings of operators I, II, III, IV

1940 ———

B: Lattice theory book

Frink, Prenowtiz: infinite-dim lattice-theoretic projective geometry

1950 ———

Kaplansky: complete modular ortholattice is a continuous geometry

Dye: morphisms of vN algebras determined by projections

1960 ———

various: development of OMLs, relations to vN algebras

1970 ———

Connes: Amenable, classification type-III, derivations

Alfsen, Schultz: Gelfand-Naimark for C*-algebras

1980 ———

Jones: Galois theory for vN algebras, subfactors, index, tower

Popa: classification of subfactors, commuting squares

1990 ———

Literally thousands of papers on factors, subfactors, planar algebras

their applications to TQFT, quantum information, etc.

Order theory, geometry, and to a lesser extent logic, played prominent roles in the development of operator algebras until about the 1970's. After ... not so much. We aim to look at more recent topics including subfactors using these traditional tools.

Background

Definition A $*$-algebra is a vector space with multiplication AB, involution A^*.

Example M_n all $n \times n$ matrices.

Example $B(H)$ all bounded operators on a Hilbert space H.

Note $B(H)$ carries a norm and weak operator topology (WOT)

Definition A C^*-algebra is a $*$-subalgebra of $B(H)$ closed in the norm top.

Definition A vN-algebra is a $*$-subalgebra of $B(H)$ closed in the WOT.

Note vN algebras $\subseteq$ C^*-algebras; vN-algebras were defined first.

Note There are abstract characterizations of C^* and vN (or W^*) algebras.

Background

Theorem A commutative C^*-algebra is isomorphic to an algebra $C(X)$ of all continuous $\mathbb{C}$-valued functions on some topological space X.

Theorem A commutative vN-algebra is isomorphic to all measurable $\mathbb{C}$-valued functions on some measure space X.

Note The study of C^*-algebras is sometimes called non-commutative topology and that of vN-algebras non-commutative measure theory.

There are many other "quantum" versions of ideas from classical mathematics such as non-commutative integration and quantum groups. The first example of this was the Birkhoff-von Neumann quantum logic.

Background

Let $\mathcal{M}$ be a vN-algebra.

Definition An element p in $\mathcal{M}$ is a projection if $p = p^2 = p^*$.

Definition $P(\mathcal{M})$ is the projections: $p^\perp = 1 - p$ and $p \leq q$ iff $pq = p = qp$.

Theorem $P(\mathcal{M})$ is a complete OML.

Theorem $\mathcal{M}$ is determined up to Jordan isomorphism by $P(\mathcal{M})$.

Note The Jordan product is $x \circ y = \frac{1}{2}(xy + yx)$.

Definition $\mathcal{M}$ is a factor if $P(\mathcal{M})$ is directly irreducible.

Definition An inclusion $\mathcal{N} \leq \mathcal{M}$ of factors is called a subfactor.

Background

Murray and von Neumann classified factors $\mathcal{M}$ using the following result.

Theorem $\mathcal{M}$ has a unique dimension function $\mathrm{D} : \mathrm{P}(\mathcal{M}) \to [0, \infty]$.

Type $\mathrm{I_n}$: D has range $\{0, \ldots, n\}$.

Type $\mathrm{II_1}$: D has range $[0, 1]$.

Type $\mathrm{II_\infty}$: D has range $[0, \infty]$.

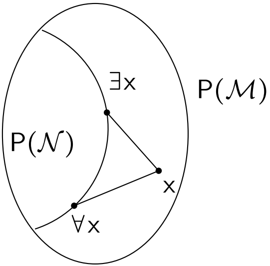
Type III: D has range $\{0, \infty\}$.

Note Type $\mathrm{I_n}$ are matrix algebras $\mathrm{M_n}$.

Note Type $\mathrm{II_1}$ factors orthocomplemented continuous geometries, but not all orthcomplemented continuous geometries are vN algebras.

Key observation

For $\mathcal{N} \leq \mathcal{M}$ a subfactor, $\mathrm{P}(\mathcal{N}) \leq \mathrm{P}(\mathcal{M})$ is a complete sub-OL.



$\exists x$ = least in $\mathrm{P}(\mathcal{N})$ above x

$\forall x$ = largest in $\mathrm{P}(\mathcal{N})$ below x

Finally, some connection to logic in this talk!

Monadic algebras

Definition A quantifier on a BA B is a map $\exists : B \to B$ where

(Q_1) $\exists 0 = 0$,

(Q_2) $p \leq \exists p$,

(Q_3) $\exists(p \vee q) = \exists p \vee \exists q$,

(Q_4) $\exists\exists p = \exists p$,

(Q_5) $\exists(\exists p)^\perp = (\exists p)^\perp$.

A monadic algebra $(B, \exists)$ is a BA B with a quantifier $\exists$.

Note: (Q_1) – (Q_5) are equivalent to (Q_1), (Q_2), (Q_6) where

(Q_6) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.

Quantum monadic algebras

Definition An OL is a bounded lattice L with unary operation $\perp$ where

(O_1) $x \wedge x^\perp = 0$

(O_2) $x \vee x^\perp = 1$

(O_3) $x \leq y \Rightarrow y^\perp \leq x^\perp$

(O_4) $x^{\perp\perp} = x$

It is an OML if it additionally satisfies

(O_5) $x \leq y \Rightarrow x \vee (x^\perp \wedge y) = y$

Monadic OLs are OLs with a quantifier $\exists$ satisfying (Q_1) – (Q_5).

Quantum monadic algebras are monadic OLs that are OMLs.

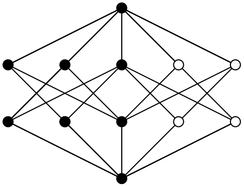
Abbreviation: q-monadic algebras.

Basic examples

Proposition If L a complete OL and $C \leq L$ is a complete subalgebra, then we have a quantifier $\exists x = \bigwedge\{c \in C : x \leq c\}$ and $(L, \exists)$ is a monadic OL.

Note: All complete monadic OLs are obtained in this way.

Example If L is a complete OML and B is a maximal Boolean subalgebra of L (such is called a block), then $B \leq L$ is a complete subalgebra. So each block of a complete OML yields a q-monadic algebra.



Examples of quantum monadic algebras

Example If $\mathcal{N} \leq \mathcal{M}$ then $\mathrm{P}(\mathcal{N}) \leq \mathrm{P}(\mathcal{M})$ yields a q-monadic algebra.

Example A von Neumann algebra $\mathcal{M}$ is specified to Jordan isomorphism by the q-monadic algebra $\mathrm{P}(\mathcal{M}) \leq \mathrm{P}(\mathrm{H})$.

Example A subfactor $\mathcal{N} \leq \mathcal{M}$ gives $\mathrm{P}(\mathcal{N}) \leq \mathrm{P}(\mathcal{M})$ a q-monadic algebra that specifies this subfactor to Jordan isomorphisms.

Slogan Subfactors are non-commutative monadic algebras.

Commuting squares

Theorem A subfactor $\mathcal{N} \leq \mathcal{M}$ has a conditional expectation $\mathrm{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$

Note This generalizes conditional expectation from measure theory.

Definition Subfactors $\mathcal{N}, \mathcal{K} \leq \mathcal{M}$ are a commuting square

$$\begin{array}{ccc} \mathcal{N} & \text{———} & \mathcal{M} \\ | & & | \\ \mathcal{N} \cap \mathcal{K} & \text{———} & \mathcal{K} \end{array}$$

if their conditional expectations $\mathrm{E}_{\mathcal{N}}$ and $\mathrm{E}_{\mathcal{K}}$ commute.

Commuting squares are well-known in subfactor theory. They are a non-commutative version of independent σ-algebras.

Theorem $\mathrm{E}_{\mathcal{N}}$ and $\mathrm{E}_{\mathcal{K}}$ commute iff the quantifiers $\exists_{\mathcal{M}}$ and $\exists_{\mathcal{K}}$ commute.

Cylindric algebras

Definition An I-dimensional cylindric algebra $(B, \exists_i, d_{i,j})$ is a BA B with a family $\exists_i$ of unary operations and $d_{i,j}$ of constants where

- (C_1) $\exists_i$ is a quantifier
- (C_2) $\exists_i \exists_j x = \exists_j \exists_i x$
- (C_3) $d_{i,j} = d_{j,i}$ and $d_{i,i} = 1$
- (C_4) if $j \neq i, k$ then $d_{i,k} = \exists_j (d_{i,j} \wedge d_{j,k})$
- (C_5) if $i \neq j$ then $\exists_i (d_{i,j} \wedge x) \wedge \exists_i (d_{i,j} \wedge x^\perp) = 0$

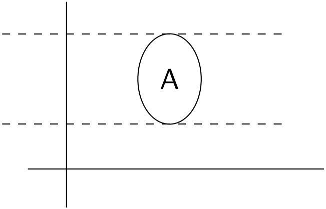
The $\exists_i$ are called cylindrifications and the $d_{i,j}$ are diagonals.

If we remove the d_{ij} we obtain a diagonal-free cylindric algebra.

(C_5) ensures $S_{ij}\, x := \exists_i (d_{ij} \wedge x)$ is a substitution endomorphism.

Cylindric algebras

The name comes from the following



$\exists_1 A$ = the cylinder generated by A

Definition Cylindric OLs are the corresponding structures with BAs replaced by OLs and quantum cylindric algebras with OMLs.

The quantum cylindric set algebra

This is closely related to Weaver's quantum logic.

Lemma For $H_1, \ldots, H_n$ Hilbert spaces, each $\mathcal{M}_i \leq B(H_1 \otimes \cdots \otimes H_n)$ is a vN subalgebra where

$$\mathcal{M}_i = \{1 \otimes A : A \in B(\bigotimes_{j \neq i} H_j)\}$$

Diagonals If all H_i are the same, diagonal D_{ij} is projection onto the subspace of the tensor power $H^{\otimes n}$ symmetric in the i^{th}, j^{th} coordinates.

Note This generalizes to infinite tensor products as well.

The quantum cylindric set algebra

Proposition For H_i $(i \in I)$ Hilbert spaces, the quantum cylindric set algebra over $\bigotimes_I H_i$ is a diagonal-free quantum cylindric set algebra.

Proposition The quantum cylindric set algebra with diagonals over the tensor power $H^{\otimes I}$ satisfies (C_1) – (C_4) but not (C_5).

Note Issues with (C_5) seem related to difficulties with substitution in Weaver's quantum predicate calculus.

Note Some issues with (C_5) are addressed by Sasaki projection. Fundamentally, substitution is far from settled — $P(\mathbb{C}^n)$ is simple!

Monadic orthoframes

Definition A relational structure $(X, \perp, R)$ is a monadic orthoframe if $\perp$ and R are binary relations on X that satisfy

(M_1) $\perp$ is symmetric and irreflexive

(M_2) R is reflexive and transitive

(M_3) for each $x \in X$, the set $R[\{x\}]^\perp$ is closed under R.

Prop $(X, \neq, R)$ is a monadic orthoframe iff R is an equivalence relation

Definition Set $(X, \perp, R)^+ = (L, \exists)$ where

1. L is the complete OL of Galois closed subsets of $(X, \perp)$.
2. $\exists A$ is the Galois closure of $R[A]$.

Monadic orthoframes

Theorem Each $(X, \perp, R)^+$ is a monadic OL. Each monadic OL is a subalgebra of such. Each complete monadic OL is isomorphic to such.

Definition $(X, \perp, (R_i)_I)$ is a diagonal-free cylindric orthoframe if

(C_1) Each $(X, \perp, R_i)$ is a monadic orthoframe

(C_2) R_i commutes with R_j for each $i, j \in I$

Theorem As above but realizing diagonal-free cylindric OLs as complex algebras of diagonal-free cylindric orthoframes.

Note There are obstacles to similar results for quantum monadic frames. It is open whether every OML can be embedded into a complete OML.

Is variation among subfactors reflected in their logical versions?

Something is funky with substitution for q-cylindric algebras.

Many examples have intrinsic probability measures. What is the appropriate logicical way to take advantage of this?

Thank You!