On the Existence of Sequent Calculi

Uniform interpolation for Lax Logic

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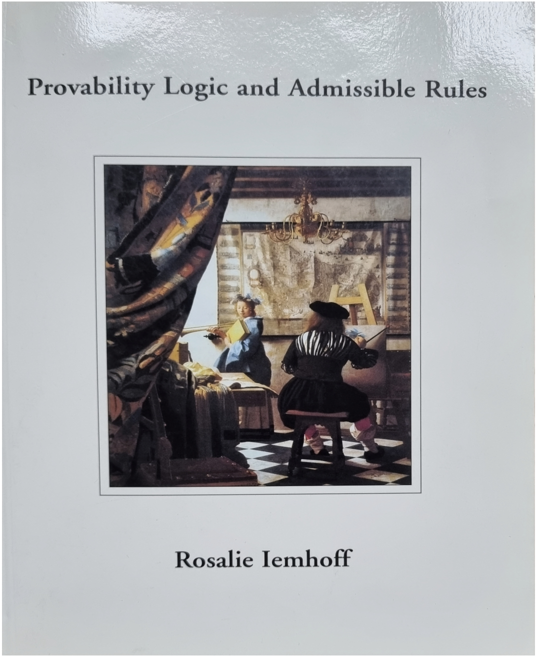


PhD thesis at the ILLC in 2001

On the Existence of Sequent Calculi

Uniform interpolation for Lax Logic

- Main Question
- Logics and Examples
- Uniform Interpolation
- Negative Results

Main Question

Numerous positive results of the form:

This logic has such and such a proof system.

Few(er) negative results of the form:

This logic does not have such and such a proof system.

Examples of negative results:

- Based on the complexity of the logic.
- On specific proof systems with a focus on cut-elimination:

  On the existence of cut-free sequent calculi –
  Belardinelli & Jipsen & Ono, later extended by Ciabattoni & Galatos & Terui.

  Labelled sequent calculi – Negri

  ⋮

Questions: Which theories have a good proof system?

Which theories lack a good proof system?

This talk: The proof systems are sequent calculi.

Good: good computational properties, e.g. finitary, cut-free or with restricted cuts, some form of subformula property.

Foundational and other strong systems: almost none (complexity).

Logics used in computer science, philosophy, artificial intelligence, and linguistics to model certain phenomena: often decidable (trading expressivity for efficiency).

For such logics the questions are relevant, and not easy to answer.

This talk:
Possible method towards a possible answer, for some classes of logics.

Def A *sequent* is an expression $(\Gamma \Rightarrow \Delta)$, where Γ and Δ are multisets of formulas, that is interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \to \bigvee \Delta)$. In intuitionistic logic, $|\Delta| \leq 1$ required.

A *sequent rule* R is an expression of the form (the S_i are sequents):

$$\frac{S_1 \dots S_n}{S_0}\ \mathrm{R}$$

Def A set of rules G is a *sequent calculus for a logic* L if: $\vdash_{\mathrm{G}} S$ iff $\models_{\mathrm{L}} I(S)$.

Sequent calculus $\mathrm{G3ip}$ for intuitionistic propositional logic IPC:

$$\Gamma, p \Rightarrow p \quad \mathrm{Ax} \quad (p \text{ an atom}) \qquad \Gamma, \bot \Rightarrow \Delta \quad \mathrm{L}\bot$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ \mathrm{R}\wedge \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \mathrm{L}\wedge$$

$$\frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1}\ \mathrm{R}\vee\ (i = 0, 1) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \mathrm{L}\vee$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ \mathrm{R}\to \qquad \frac{\Gamma, \varphi \to \psi \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \mathrm{L}\to$$

An important rule, the Cut rule, is admissible in $\mathrm{G3ip}$:

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}$$

Questions: Which logics have a good sequent calculus?

Which logics lack a good sequent calculus?

General aim:
Provide (classes of) logics with good sequent calculi, for a specific notion of "good".

Establish for classes of logics $\mathcal{L}$ and classes of sequent calculi $\mathcal{G}$ that no logic in $\mathcal{L}$ has a sequent calculus in $\mathcal{G}$.

Examples of $\mathcal{L}$: classes of modal, intermediate, substructural, intuitionistic modal logics.

Examples of $\mathcal{G}$: classes of cut-free sequent calculi with some additional properties.

This talk: the second aim.

Key: Connection between the structural properties of a sequent calculus and the (uniform) interpolation of its logic.

Logics and Examples

Def The languages of our logics consists of atoms, connectives $\wedge, \vee, \neg, \rightarrow$, and possibly modal operators $\Box, \bigcirc$.

The logics are classical and intuitionistic (normal or non-normal) modal and intermediate logics.

Running example:

Def The language $\mathcal{L}_{\mathrm{PLL}}$ of Lax Logic has one unary modal operator $\bigcirc$.

Lax Logic PLL: IPC plus the axioms

$$\varphi \rightarrow \bigcirc\varphi \qquad \bigcirc\bigcirc\varphi \rightarrow \bigcirc\varphi \qquad \bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi).$$

PLL appears in:

- hardware verification ($\bigcirc\varphi \sim \varphi$ *holds under some constraint*).
- algebraic logic ($\bigcirc$ is a *nucleus*, a certain modal operator on Heyting Algebras).
- type theory ($\bigcirc$ is a type constructor).

The sequent calculus GLL for PLL consists of $\mathrm{G4i}$ plus the rules

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \bigcirc\varphi}\ R\bigcirc \qquad \frac{\Gamma, \varphi \Rightarrow \bigcirc\psi}{\Gamma, \bigcirc\varphi \Rightarrow \bigcirc\psi}\ L\bigcirc$$

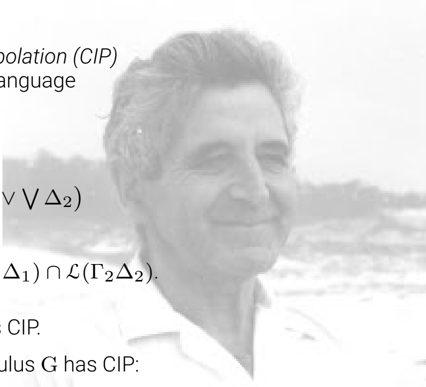
$\mathrm{G4i}$ is a terminating variant of the sequent calculus $\mathrm{G3ip}$ for IPC (Dyckhoff 1997).

calculus to property

Sequent calculi are used in many proofs, e.g. in proofs of decidability, interpolation, disjunction property, Herbrand's Theorem …

Example:

Def A propositional (modal) logic L has *Craig interpolation (CIP)* if whenever $\vdash \varphi \to \psi$ there is a χ in the common language $\mathcal{L}(\varphi) \cap \mathcal{L}(\psi)$ such that $\vdash \varphi \to \chi$ and $\vdash \chi \to \psi$.



Calculus G has *sequent interpolation (SIP)* if

$$\vdash_{\mathrm{G}} \Gamma_1\Gamma_2 \Rightarrow \Delta_1\Delta_2 \quad (\bigwedge \Gamma_1 \wedge \neg \bigvee \Delta_1 \to \neg \bigwedge \Gamma_2 \vee \bigvee \Delta_2)$$

implies

$\vdash_{\mathrm{G}} \Gamma_1 \Rightarrow \chi, \Delta_1$ & $\vdash_{\mathrm{G}} \Gamma_2, \chi \Rightarrow \Delta_2$ for a $\chi \in \mathcal{L}(\Gamma_1\Delta_1) \cap \mathcal{L}(\Gamma_2\Delta_2)$.

Lemma If G is a calculus for L with SIP, then L has CIP.

Proof-theoretic proof that a given logic L with calculus G has CIP:

1. The axioms in G have interpolants.
2. The rules $S_1 \dots S_n / S$ in G are closed under interpolation: if the S_i have interpolants, then S has an interpolant.
3. Therefore G has SIP, and thus L has CIP.

Uniform Interpolation

Def A logic L has *uniform interpolation (UIP)* if the interpolant depends only on the premise or the conclusion: For all atoms p and formulas φ there are formulas, denoted $\exists p\varphi$ and $\forall p\varphi$, in the language of L that do not contain p or any variable not in φ, such that for all ψ not containing p:

$$\vdash \psi \to \varphi \text{ iff } \vdash \psi \to \forall p\varphi \quad \vdash \varphi \to \psi \text{ iff } \vdash \exists p\varphi \to \psi.$$

$$\vdash \varphi \to \exists p\varphi \quad \vdash \forall p\varphi \to \varphi.$$

Ex $\vdash \varphi \to \exists p\varphi \quad \vdash \forall p\varphi \to \varphi \quad \exists q((p \to q) \wedge \neg q) = \neg p.$

Def A sequent calculus G has *uniform sequent interpolation (USIP)* if for any sequent $S = (\Gamma \Rightarrow \Delta)$ there exist formulas $\exists pS$ and $\forall pS$ s. t. for all p-free Π, Σ:

$$\vdash \Gamma \Rightarrow \exists pS, \Delta \quad \vdash \Gamma, \forall pS \Rightarrow \Delta$$
$$\vdash \Gamma, \Pi \Rightarrow \Delta, \Sigma \text{ implies } \vdash \Pi, \exists pS \Rightarrow \forall pS, \Sigma.$$

Lemma If G is a calculus for logic L with USIP, then L has UIP.

Thm (Pitts '92)
IPC has uniform interpolation. (inspiration for our approach)

Thm (Shavrukov '94)
GL has uniform interpolation.

Thm (Ghilardi & Zawadowski '95)
K has uniform interpolation. S4 does not.

Thm (Bilkova '06)
KT has uniform interpolation. K4 does not.

Thm (Maksimova '77, Ghilardi & Zawadowski '02)
There are exactly seven intermediate logics with (uniform) interpolation:

$$\mathrm{IPC}, \mathrm{Sm}, \mathrm{GSc}, \mathrm{LC}, \mathrm{KC}, \mathrm{Bd}_2, \mathrm{CPC}.$$

Thm (Maksimova '91)
Among the normal extensions of S4 there are at least 31 and at most 49 logics with interpolation. Exactly 7 normal extensions of Grz have interpolation.

Pitts uses a terminating sequent calculus G4i for IPC.
(developed independently by Dyckhoff and Hudelmaier in '92)

Thm (I. 2021) Lax Logic has UIP.

Prf idea:

Def The sequent calculus GLL for PLL consists of $\mathrm{G4i}$ plus the rules

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \bigcirc\varphi}\ R\bigcirc \qquad \frac{\Gamma, \varphi \Rightarrow \bigcirc\psi}{\Gamma, \bigcirc\varphi \Rightarrow \bigcirc\psi}\ L\bigcirc$$

$\mathrm{G4i}$ is a terminating variant of the standard single-conclusion sequent calculus $\mathrm{G3ip}$ without structural rules for IPC (Dyckhoff 1997).

(Thm (Fairtlough & Mendler 1994)
There is an extension of LJ that is an analytic calculus for PLL in which the cut rule is admissible.)

Define for any sequent S formulas $\forall p S$ and $\exists p S$ in terms of the rules of the calculus and show that they are uniform sequent interpolants for S, using induction on the size of sequents and on the length of proofs in GLL.

(All complicated details omitted.)

$\dashv$

Negative Results

To show that no logic in a class of logics $\mathcal{L}$ has a calculus $\mathrm{G} \in \mathcal{G}$:

1. Prove that all calculi in $\mathcal{G}$ have USIP.
2. Prove that all logics in $\mathcal{L}$ do not have UIP.
3. Conclude that no logic in $\mathcal{L}$ can have a calculus in $\mathcal{G}$.

Question
For which classes $\mathcal{L}$ of modal and intermediate logics and which classes $\mathcal{G}$ of calculi can we prove 1 and 2?

Guiding example
The proof that PLL has UIP. Key properties of GLL in the proof of USIP:

One GLL is a balanced calculus.

Two The nonmodal rules of GLL are semi-analytic rules.

(nex slide)

Def A well-founded order on sequents is *reductive* if

- all proper subsequents of a sequent come before that sequent;
- whenever all formulas in S occur boxed in S', then $S \prec S'$;
- $(\Gamma, \varphi \Rightarrow \Delta) \prec (\Gamma, q \to \varphi \Rightarrow \Delta)$ for all $\Gamma, \Delta, \varphi, q$.

Def A calculus G is *terminating* with respect to order $\prec$ if

- G is finite;
- for any S there are at most finitely many instances of rules with conclusion S;
- in any rule of the calculus the premises come before the conclusion in order $\prec$.

G is *reductive* if it is terminating with respect to a reductive order.

Note Many well-known sequent calculi are reductive with respect to the Manna and Dershowitz multiset-ordering of sequents.

Def A calculus for an intermediate logic is *balanced* if it is reductive and Cut and Left Weakening are admissible in it.

Def A calculus for an (intuitionistic) modal logic is *balanced* if it is reductive, Cut and Left Weakening are admissible in it, and it satisfies a requirement on the interaction between different modal rules (omitted).

Def A *semi-analytic* rule is of one of the two forms

$$\frac{[[\Gamma_i, \overline{\psi}_{ik} \Rightarrow \overline{\chi}_i]_k]_i \quad [[\Pi_h, \overline{\theta}_{hl} \Rightarrow \Delta_h]_l]_h}{\Gamma_1, \ldots, \Gamma_m, \Pi_1, \ldots, \Pi_n, \varphi \Rightarrow \Delta_1, \ldots, \Delta_n} \qquad \frac{[[\Gamma_i, \overline{\psi}_i \Rightarrow \chi_{ik}]_k]_i}{\Gamma_1, \ldots, \Gamma_m \Rightarrow \varphi}$$

(all atoms in $\overline{\psi}_i, \overline{\chi}_i, \overline{\theta}_j$ occur in φ and $|\cup_i \Delta_i| \leq 1$)

Def The definition of *modal semi-analytic rule* is omitted. Standard modal rules for K, $\mathrm{K4}$, KT, …are of that form.

Ex All rules of $\mathbf{G3cp}$ are semi-analytic.

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi} \quad \frac{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}{\Gamma, \varphi, \psi \Rightarrow \Delta}$$

Note Many well-known calculi for intermediate and (intuitionistic) modal logics are balanced and consist mostly of semi-analytic rules.

Positive result:

Thm (I. 2017, Jalali & Tabatabai 2018)
Any intermediate logic with a balanced calculus that is an extension of $\mathrm{G4i}$ by semi-analytic rules has uniform interpolation.

Prf Generalization of the method for PLL.

Cor (Pitts 1992)
IPC has uniform interpolation.

Negative result:

Cor All except seven intermediate logics have no such calculi.

Prf Only seven intermediate logics have UIP.

Cor Any balanced calculus for a logic without UIP contains some rules that are not semi-analytic.

Positive results:

Thm (I. 2017)
KD(Pattinson), iK and iKD have uniform interpolation.

Thm (I. 2021)
Lax Logic PLL has uniform interpolation.

Thm (I., Jalali & Tabatabai 2021)
Any normal intuitionistic modal logic with a balanced calculus that consists of semi-analytic (modal) rules has uniform interpolation.

Thm (I., Jalali & Tabatabai 2022)
The non-normal modal and conditional logics E, M, CE, and CM (and some more) have uniform Lyndon interpolation. (same method)

Negative results:

Cor Any balanced calculus for a logic without UIP contains some rules that are not (modal) semi-analytic.

Cor The modal logics $\mathrm{K4}$ and $\mathrm{S4}$ do not have UIP and therefore no balanced calculus consisting of semi-analytic (modal) rules.

Ongoing:

- Extensions to non-normal (intuitionistic) modal logics.
- Similar approach with disjunction property instead of UIP (Jalali & Tabatabai).
- Investigate the tightness between the termination of a calculus and UIP.

Conclusions:

For substructural, intermediate, and (intuitionistic) modal logics:

- uniform proof-theoretic way to prove UIP;
- numerous applications of the above to concrete logics, such as Lax Logic;
- a method to prove that certain logics cannot have certain calculi.

Finis