Quantified Reflection Calculus towards the polymodal case

Ana de Almeida Borges **Joost J. Joosten**

Universitat de Barcelona

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Formalised provability and applications

- Provability is a central notion in logic and metamathematics
- For theories like PA we can write a Σ_1 predicate $\Box_{\mathsf{PA}}(\cdot)$ such that

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Theorem

The $\Box_{\mathsf{PA}}(\cdot)$ predicate is Σ^0_1-complete. That is, for each c.e. set A, there is an arithmetical formula $\rho_A(x)$ such that

$$A = \{n \in \mathbb{N} \mid \mathbb{N} \models \Box_{\mathsf{PA}}\Big(\rho_A(n)\Big)\}.$$

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  If PA $\vdash \Box_{\mathsf{PA}}(\ulcorner A \urcorner) \to A$, *then* PA $\vdash A$, *for any* PA *formula* A

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- Characterise all provably structural properties in two steps
  - $\mathcal{L}_\Box$ with $\mathrm{Form}_\Box := \bot \mid \mathrm{Prop} \mid \mathrm{Form}_\Box \to \mathrm{Form}_\Box \mid \Box \mathrm{Form}_\Box$
  - Define a denotation of $\mathcal{L}_\Box$ formulas inside the $\mathcal{L}_{\mathrm{PA}}$ formulas

Arithmetical realizations

An arithmetical realization is any function $(\cdot)^\star$ taking:

$$\text{formulas in } \mathcal{L}_\Box \to \text{sentences in } \mathcal{L}_{\mathsf{PA}}$$
$$\text{propositional variables} \to \text{arithmetical sentences}$$
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Clearly, for any realization $(\cdot)^\star$ we have for example

$$\mathsf{PA} \vdash \Big(\Box(p \to q) \to (\Box p \to \Box q)\Big)^\star$$

since

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}}(p^\star \to q^\star) \to (\Box_{\mathsf{PA}} p^\star \to \Box_{\mathsf{PA}} q^\star)$$

regardless of $(\cdot)^\star$

The Provability Logic of a Theory

- For a c.e. theory T we define

$$\mathrm{PL}(T) := \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } T \vdash (\varphi)^\star\}$$

- Here $(\cdot)^\star$ is as before, but now mapping $\Box$ to $\Box_T$

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A candidate

- GL is the normal modal logic with axioms
  - All classical logical tautologies in $\mathcal{L}_\Box$ like $\Box p \vee \neg\Box p$, etc.
  - All distributions axioms: $\Box(A \to B) \to (\Box A \to \Box B)$,
  - All Löb axioms: $\Box(\Box A \to A) \to \Box A$.
- and rules
  - Modus Ponens $\dfrac{A \to B \quad A}{B}$,
  - Necessitation $\dfrac{A}{\Box A}$.

Solovay's Theorem

Theorem (Solovay, 1976)

Let $\varphi \in \mathcal{L}_\Box$. *Then:*

$$\mathsf{GL} \vdash \varphi$$
$$\Updownarrow$$
$$\mathsf{PA} \vdash (\varphi)^\star \textit{ for any arithmetical realization } (\cdot)^\star$$

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Thus, even though PL(PA) is *prima facie* of complexity Π^0_2, it allows for a decidable description

$$\mathsf{GL} = \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

of complexity PSPACE.

True provability logic

- $\mathsf{PA} \nvdash \Box_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner) \to 0 = 1$
- $\mathbb{N} \models \Box_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \varphi$ for whatever sentence φ

For a c.e. theory T we define

$$\mathsf{TPL}(T) := \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathbb{N} \models (\varphi)^\star\}$$

A priori, complexity above true arithmetic.
However,

$$\mathsf{TPL}(\mathsf{PA}) = \mathsf{GLS}.$$

Here GLS is axiomatised by all theorems of GL and all reflection axioms $\Box A \to A$ with MP as the only rule.

Solovay for quantified modal logic?

Let $\mathcal{L}_{\Box,\forall}$ be the language of relational quantified modal logic:

$$\top\text{, relation symbols, boolean connectives, } \forall x\text{, and } \Box$$

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$$\text{formulas in } \mathcal{L}_{\Box,\forall} \to \text{formulas in } \mathcal{L}_{\mathsf{PA}}$$

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For a c.e. theory T we define

$$\mathsf{QPL}(T) := \{\varphi \in \mathcal{L}_{\Box,\forall} \mid \text{for any } (\cdot)^\bullet, \text{ we have } T \vdash (\varphi)^\bullet\}$$

and

$$\mathsf{TQPL}(T) := \{\varphi \in \mathcal{L}_{\Box,\forall} \mid \text{for any } (\cdot)^\bullet, \text{ we have } \mathbb{N} \models (\varphi)^\bullet\}$$

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Example: $\Box\forall x P(x) \to \forall x \Box P(\dot{x})$

A Degenerate Quantified Provability Logic

If we define $\mathsf{QL}(T) = \{\varphi \in \mathcal{L}_{\mathsf{fol}} \mid \text{for any } (\cdot)^\bullet, \text{ we have } T \vdash (\varphi)^\bullet\}$, then it is not hard to see that $\mathsf{CQC} = \mathsf{QL}(\mathsf{PA})$.

Proof:

- $\subseteq$ if $\pi \vdash_{\mathsf{CQC}} \varphi$, then also $\pi^\bullet \vdash_{\mathsf{CQC}} \varphi^\bullet$, whence $\pi^\bullet \vdash_{\mathsf{PA}} \varphi^\bullet$
- $\supseteq$ Henkin construction in arithmetic

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$$\mathsf{QPL}(\mathsf{PA} + \mathsf{Incon}(\mathsf{PA})) = \mathsf{CQC} + \Box\bot$$

Negative results

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Theorem (Vardanyan, 1986 and McGee, 1985)

$$\{\textit{closed } \varphi \in \mathcal{L}_{\Box,\forall} \mid \textit{for any } (\cdot)^\bullet, \ \textit{we have } \mathsf{PA} \vdash (\varphi)^\bullet\}$$

is Π^0_2*-complete. Thus it is not recursively axiomatisable.*

Theorem (Artemov, 1985)

TQPL(PA) *is not arithmetical.*

Theorem (Vardanyan, 1985)

TQPL(PA) *is* Π^0_1 *complete in true arithmetic.*

Artemov's Lemma

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$\mathcal{L}_{PA}$

F

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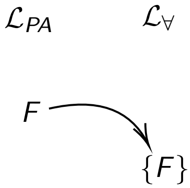
- Replace arithmetical symbols $0, +1, +, \times, =$ with predicates Z, S, A, M, E, obtaining $\{F\} \in \mathcal{L}_{\forall}$



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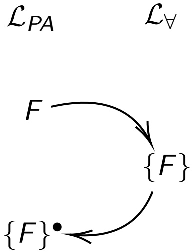

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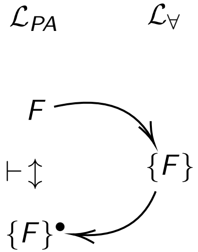
When are F and $\{F\}^{\bullet}$ equivalent over PA?



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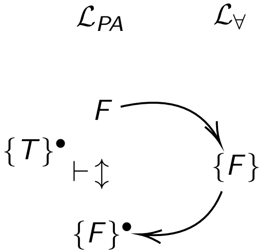

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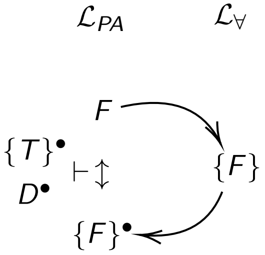

$$
\begin{aligned}
D := {} & \Diamond \top \wedge \\
& \forall x\,(Z(x) \to \Box Z(x)) \wedge \forall x\,(\neg Z(x) \to \Box \neg Z(x)) \wedge \\
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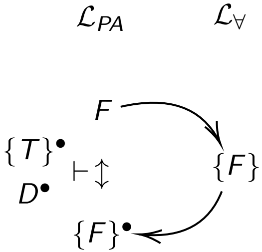

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- Under $\{T\}^\bullet$ to get arithmetical axioms...
- ... and under $D^\bullet$ to get recursive $A^\bullet$ and $M^\bullet$
- By Tennenbaum's Theorem the model induced by $(\cdot)^\bullet$ is standard, hence $\mathbb{N} \models S \iff (\{T\} \wedge D \to \{S\}) \in \mathsf{TQPL(PA)}$

$$D := \Diamond\top \wedge$$
$$\forall x\,(Z(x) \to \Box Z(x)) \wedge \forall x\,(\neg Z(x) \to \Box\neg Z(x)) \wedge$$
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One easily sees that $\mathsf{QPL}(\mathsf{PA} + \Box_{\mathsf{PA}}\bot)$ is r.e., but it seems that $\mathsf{QPL}(\mathsf{PA} + \Box_{\mathsf{PA}}\Box_{\mathsf{PA}}\bot)$ is also Π^0_2-complete.

Theorem (Visser, de Jonge, 2006)

$\mathsf{QPL}(T)$ *is* Π^0_2 *complete for any* Σ_1*-sound theory* T *extending* EA.

Archive for Mathematical Logic 2006: No Escape from Vardanyan's

Planning an escape

Restrict $\mathcal{L}_{\Box,\forall}$ to the strictly positive fragment $\mathcal{L}_{\Diamond,\forall}$:

$$\text{Terms} ::= \text{Variables} \mid \text{Constants}$$

$$\mathcal{L}_{\Diamond,\forall} ::= \top \mid \text{relation symbols applied to Terms} \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \Diamond\varphi$$

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Prove arithmetical soundness and completeness for QRC$_1$:

$$\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \text{for any } (\cdot)^*, \text{ we have } \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$$

Old escapes

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- Yavorski, add $\Box A \to \Box \forall x A$

Four current trends

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$$A \vdash_{\mathrm{sp}(L)} B \quad \iff \quad L \vdash A \to B$$

- Reflection Calculi: replace the realisation $p^\star$ by a (simple) axiomatisation of an arbitrary theory
  (instead of mapping $p^\star$ to an arbitrary sentence)
- Polymodal provability logics: GLP is a polymodal version of GL, with $[0], [1], \ldots$ as modalities
  - Decidability is PSPACE-complete
  - RC is the strictly positive fragment of GLP, with statements of the form $\varphi \vdash \psi$, where φ, ψ are in the language built from $\top$, p, $\wedge$, $\langle 0 \rangle, \langle 1 \rangle, \ldots$
  - E.g. $\langle 1 \rangle p \vdash \langle 0 \rangle p$
  - Decidability is in PTIME

Four current trends

- Strictly positive fragments of modal logics (Zakharyashev, Wolter, *et al.*)

$$A \vdash_{\mathsf{sp}(L)} B \quad \Longleftrightarrow \quad L \vdash A \to B$$

- Reflection Calculi: replace the realisation $p^\star$ by a (simple) axiomatisation of an arbitrary theory
  (instead of mapping $p^\star$ to an arbitrary sentence)
- Polymodal provability logics: GLP is a polymodal version of GL, with $[0], [1], \ldots$ as modalities
  - Decidability is PSPACE-complete
  - RC is the strictly positive fragment of GLP, with statements of the form $\varphi \vdash \psi$, where φ, ψ are in the language built from $\top$, p, $\wedge$, $\langle 0 \rangle, \langle 1 \rangle, \ldots$
  - E.g. $\langle 1 \rangle p \vdash \langle 0 \rangle p$
  - Decidability is in PTIME
- Workshop on Decidable Fragments of First-order Modal Logic, Affiliated workshop of LICS 2022

QRC$_1$: Axioms and rules

$$\varphi \vdash \top \qquad \varphi \wedge \psi \vdash \varphi$$

$$\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

QRC$_1$: Axioms and rules

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$x \notin \mathsf{fv}\,\varphi$ t free for x in φ

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$$\frac{\varphi[x \leftarrow t] \vdash \psi}{\forall x\, \varphi \vdash \psi}$$

t free for x in φ

$$\frac{\varphi \vdash \psi}{\varphi[x \leftarrow t] \vdash \psi[x \leftarrow t]}$$

t free for x in φ and ψ

$$\frac{\varphi[x \leftarrow c] \vdash \psi[x \leftarrow c]}{\varphi \vdash \psi}$$

c not in φ nor ψ

Some provable and unprovable statements

$$\Diamond \forall x\, \varphi \vdash \forall x\, \Diamond \varphi$$

$$\forall x\, \Diamond \varphi \nvdash \Diamond \forall x\, \varphi$$

$$\frac{\varphi \vdash \psi[x \leftarrow c]}{\varphi \vdash \forall x\, \psi}$$

x not free in φ and c not in φ nor ψ

Recall that RC_ω allows for ordinal notations up to ε_0 and that it caters Π^0_1 ordinal analyses.

Can be extended to RC_Λ.

Arithmetical semantics

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$:

$$\text{formulas in } \mathcal{L}_{\Diamond,\forall} \to \text{axiomatisations of c.e. theories in } \mathcal{L}_{\mathsf{PA}}$$
$$\text{variables } x_i \to \text{variables } y_i$$
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(\top)^* &:= \tau_{\mathsf{PA}}(u) \\
(S(x,c))^* &:= \sigma(y,z,u) \vee \tau_{\mathsf{PA}}(u) \;\; \text{with } \sigma \in \Sigma_1 \\
(\psi(x,c) \wedge \delta(x,c))^* &:= (\psi(x,c))^* \vee (\delta(x,c))^* \\
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(\varphi(x,c) \vdash \psi(x,c))^* &:= \forall \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)
\end{aligned}$$

Arithmetical soundness

Theorem (Arithmetical soundness)

$$\mathsf{QRC}_1 \subseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \textit{ we have } \mathsf{PA} \vdash \forall\, \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)\}$$

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By induction on the QRC$_1$-proof. Here is the case of $\Diamond\Diamond\varphi \vdash \Diamond\varphi$:

- Pick any $(\cdot)^*$, reason in T, and let θ, y, z be arbitrary
- Assume $\Box_{(\Diamond\varphi)^*}\theta$
- Then $\Box_{\mathsf{PA}}(\mathsf{Con}_{\varphi^*}(\top) \to \theta)$
- By provable Σ_1-completeness, $\Box_{\mathsf{PA}}(\mathsf{Con}_{\mathsf{PA}}(\mathsf{Con}_{\varphi^*}(\top)) \to \mathsf{Con}_{\varphi^*}(\top))$
- Then $\Box_{\mathsf{PA}}(\mathsf{Con}_{\mathsf{PA}}(\mathsf{Con}_{\varphi^*}(\top)) \to \theta)$
- We conclude $\Box_{(\Diamond\Diamond\varphi)^*}\theta$
- Σ_1-collection is needed for $\dfrac{\varphi \vdash \psi}{\varphi \vdash \forall x \psi}$ with $x \notin \varphi$

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Where T is a sound r.e. theory extending $\mathrm{I}\Sigma_1$.

Adapt Solovay's completeness proof:

- Need Kripke completeness for QRC$_1$

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- ...

Relational models

Kripke models where:

- each world w is a first-order model with a finite domain D
- the domain D is the same for every world
- each constant symbol c and relational symbol S has a denotation at each world
- there is a transitive relation R between worlds
- constants have the same denotation at every world
- the denotation of a relation symbol depends on the world

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- constants have the same denotation at every world
- the denotation of a relation symbol depends on the world
- we use assignments $g : \text{Variables} \to D$ to interpret variables
- we abuse notation and define $g(c) := \text{denotation}(c)$ for all assignments g and constants c

Satisfaction

Let g be a w-assignment.

$$\mathcal{M}, w \Vdash^g S(t,u) \iff \langle g(t), g(u)\rangle \in \text{denotation}_w(S)$$

$\mathcal{M}, w \Vdash^g \Diamond\varphi \iff$

there is a world v such that wRv and $\mathcal{M}, v \Vdash^g \varphi$

$\mathcal{M}, w \Vdash^g \forall x\, \varphi \iff$

for all assignments $h \sim_x g$, we have $\mathcal{M}, w \Vdash^h \varphi$

Relational soundness

Theorem (Relational soundness)

If $\varphi \vdash \psi$, then for any model $\mathcal{M}$, world w, and assignment g:

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Countermodels with arbitrarily large domains are needed.

$$\forall x, y\, S(x, x, y) \wedge \forall x, y\, S(x, y, x) \wedge \forall x, y\, S(y, x, x) \vdash \forall x, y, z\, S(x, y, z)$$

is unprovable in QRC$_1$, but satisfied by every world with at most two domain elements.

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Can be extended to any n: with S n-ary, let φ be the conjunction of the $n(n-1)/2$ formulas of the form $\forall x_0, \ldots, x_{n-2}\, S(\ldots, x_0, \ldots, x_0, \ldots)$. Now φ does not entail $\psi := \forall x_0, \ldots, x_{n-1}\, S(x_0, \ldots, x_{n-1})$. Worlds with $\leq n-1$ elements that satisfies φ must also satisfy ψ.

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If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and an assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \text{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Since QRC$_1$ has the finite model property (finite number of worlds with finite constant domain), it is decidable.

Proving relational completeness

- Given $\varphi \not\vdash \psi$, build a counter-model
- The standard is to use term models: each world is the set of formulas true at that world
- We also want to know which formulas are *not* true at given worlds
- Our worlds are pairs of "positive" (true) and "negative" (false) formulas:

$$w = \langle w^+, w^- \rangle \qquad \text{e.g. } \langle \{\varphi\}, \{\psi\} \rangle$$

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- Worlds should be *well-formed* pairs though...

Well-formed pairs

Let Λ be a set of formulas and p be a pair.

- $\Gamma \vdash \delta$ is shorthand for $(\bigwedge_{\gamma \in \Gamma} \gamma) \vdash \delta$
- p is *closed* if every formula in p is closed

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- p is Λ-*well-formed* if it is closed, Λ-maximal, consistent and fully witnessed

Building a world from an incomplete pair

- Let Λ be a finite set of closed formulas
- Let C be a finite set of constants containing the constants in Λ and some new constants
- Let Λ_C be the closure under (closed) subformulas of Λ, and such that if $\forall x\, \varphi \in \Lambda_C$, then for every $c \in C$ we have $\varphi[x\leftarrow c] \in \Lambda_C$
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Method

- Some formulas in Λ_C are consequences of p^+, and thus must be added to w^+ to preserve consistency
- We put all the other formulas of Λ_C in p^-

This Method works!

Lemma

If $|C| > 2$*(max. constant count in* Λ*)* $+ 2$*(max.* $\forall$*-depth of* Λ*) and* p^+ *is a singleton, the Method produces a* Λ_C*-well-formed pair* w*.*

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$$\text{there is some } c \in C \text{ s.t. } c \text{ doesn't appear in } \forall x\, \varphi \text{ nor } p^+$$

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Building a counter-model

- Start with $\varphi \nvdash \psi$ (both closed)
- Build a (well-formed!) world w by extending $p := \langle\{\varphi\}, \{\psi\}\rangle$ (with $\Lambda := \{\varphi, \psi\}$ and C large enough for Λ)
- Let the domain be the set of constants C
- Let the denotation of relation symbols at w correspond to their membership in w^+

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- Let the domain be the set of constants C
- Let the denotation of relation symbols at w correspond to their membership in w^+
- If $\Diamond\chi \in w^+$, create a new world v_χ seen from w by Λ_C-completing

$$\langle \{\chi\}, \{\delta, \Diamond\delta \mid \Diamond\delta \in w^-\} \cup \{\Diamond\chi\} \rangle$$

- Define the domain and the denotation at v_χ like with w
- Repeat until all $\Diamond$-formulas are witnessed

Putting it together

Lemma (Truth lemma)

Let $\mathcal{M}$ be the counter-model we just built. Then for any world w, assignment g, and formula $\chi^g \in \Lambda_C$:

$$\mathcal{M}, w \Vdash^g \chi \iff \chi^g \in w^+,$$

where χ^g is χ with every free variable x replaced by $g(x)$.

Theorem (Relational completeness)

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  - $T \vdash \bigwedge_{i \neq j}(\lambda_i \to \neg\lambda_j)$

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  - $T \vdash \bigwedge_{iRj} (\lambda_i \to \Diamond\lambda_j)$

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- Contradiction!

Corollaries

Theorem (Fragment of QPL(PA))

$$\varphi \vdash_{\mathsf{QRC_1}} \psi \quad \Leftrightarrow \quad (\varphi \to \psi) \in \mathsf{QPL(PA)}$$

Theorem (Positive fragment)

Let φ *and* ψ *be* QRC$_1$ *formulas (no constants) and let* QS *be any logic between* QK4 *and* QGL*. Then* $\varphi \vdash_{\mathsf{QRC_1}} \psi$ *if and only if* $\mathsf{QS} \vdash \varphi \to \psi$.

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Heyting Arithmetic

Theorem

$$\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \; \textit{we have } \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$$

- Soundness also works for HA

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- PA is provably Π^0_2 conservative over HA
- Complexity of unprovable substitutions using Solovay is Σ_2
- This seems to leave room for generalising to HA

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- Recall that PL(HA) is a long-standing open problem

In summary

- There is no quantified provability logic with $\mathcal{L}_{\Box,\forall}$

QRC$_1$:

- quantified, strictly positive provability logic with $\mathcal{L}_{\Diamond,\forall}$
- decidable
- sound and complete w.r.t. relational semantics (with constant domain models!)
- sound and complete w.r.t. arithmetical semantics
- for all sound r.e. theories extending $\mathrm{I}\Sigma_1$

Forthcoming research

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- Strictly positive fragments of modal mu calculus
- Modal mu calculus to capture infinite dynamics in GLP (Reduction Property, Reflexive points in RC models, etc.)

Thank you

Further Reading I

S.N. Artemov (1985)
Nonarithmeticity of truth predicate logics of provability.
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