Definable Henselian valuations in positive residue characteristic

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§ Motivation

Let \( p \) be a prime, \( \mathbb{Z} \subseteq \mathbb{Q} \). Define:

\[ \psi(a) = \max \{ n \in \mathbb{N} : p^n \text{ divides } a \} \]

An extended \( \psi \) to \( \mathbb{Q} \):

\[ \psi\left( \frac{a}{b} \right) = \psi(a) - \psi(b) \in \mathbb{Z}, \quad \psi(0) = \infty \]

\( \psi \) induces a metric on \( \mathbb{Q} \):

\[ d(x,y) = p^{-\psi(x-y)} \in \mathbb{R}_0^+ \]

(Counter-) intuition: \( x, y \in \mathbb{Q} \) are close to each other if

\[ \psi(x-y) \text{ is small} \]

Completing yields \( (\mathbb{Q}_p, \psi) \), the field of \( p \)-adic numbers

\( \psi : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{ \infty \} \)

Special subclass, the \( p \)-adic integers

\[ \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : \psi(x) = 0 \} \]

Fact (J. Robinson) \( p > 2 \)

\[ \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : 3y \in \mathbb{Q}_p (1 + px^2 - y^2) \} \]

i.e. \( \mathbb{Z}_p \) is \( \mathbb{Z} \)-\( \mathbb{Q} \)-definable in \( \mathbb{Q}_p \)

§ Valuations

Def. An ordered abelian group \( (\Gamma, +, \leq) \) is a group \( (\Gamma, +) \) together with a total order \( \leq \) on \( \Gamma \) such that + and \( \leq \) are compatible,

\[ a \leq b \implies ac \leq bc \text{ for all } a, b, c \in \Gamma \]

We introduce a symbol \( \infty \) with the usual rules, e.g.

\[ \forall a \in \Gamma : a \leq \infty, \quad a + a = a_0 = \infty \]

\[ \infty + a = \infty, \quad \infty + \infty = \infty \]

Examples

* \( (\mathbb{Z}, \leq) \), \( (\mathbb{Q}^+, \leq) \), \( (\mathbb{R}, \leq) \)

* in general subgroups of \( \mathbb{R} \)

* \( (\mathbb{Z} \times \mathbb{Z}, \leq) = (\mathbb{Z}, \leq) \times (\mathbb{Z}, \leq) \)

* \( (\mathbb{Q}, \leq) \) is \( \mathbb{Z} \)-linearly ordered by \( \leq \)

* complementation addition:

\[ (a_0, b_0) \leq (a_2, b_2) \iff a_1 = a_2 \text{ or } (a_1 - a_2 \text{ and } b_1 = b_2) \]
Def. Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a surjective map $v : \mathbb{K} \to \Gamma \cup \{\infty\}$ where $(\Gamma, +, \cdot)$ is an ordered abelian group and such that:

1. $v(x) = \infty \iff x = 0$ for all $x \in \mathbb{K}$
2. $v(xy) = v(x) + v(y)$ for all $x, y \in \mathbb{K}$
3. $v(xy) \geq \min\{v(x), v(y)\}$ for all $x, y \in \mathbb{K}$

We say $(\mathbb{K}, v)$ is a valued field.

$\Gamma$ is called the value group of $(\mathbb{K}, v)$, also write $\mathbb{K}=v^{-1}(\Gamma)$. $\Gamma$

Recall the intuition: "close to 0 $\iff$ big valuation.

Examples:
- $(\mathbb{Q}, v_p)$, $(\mathbb{Q}_p, v_p)$ have value group $\mathbb{Z}$
- $(\mathbb{K}, v_{\text{triv}})$, $v_{\text{triv}} : \mathbb{K} \to \mathbb{Q} \cup \{\infty\}$
  $v_{\text{triv}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0 \end{cases}$

Def. $(\mathbb{K}, v)$ valued field:
- $\mathcal{O}_v = \{x \in \mathbb{K} : v(x) = 0\}$, the valuation ring
- is a local ring with unique maximal ideal $m_v = \{x \in \mathbb{K} : v(x) > 0\}$
- $\mathbb{K}_v = \mathcal{O}_v/m_v$, the residue field

Thus a valued field has three characteristics: $\text{char}(\mathbb{K})$ & $\text{char}(\mathbb{K}_v)$
- $\text{char}(\mathbb{K}) = \text{char}(\mathbb{K}_v) = 0$ "equicharacteristic 0"
- $\text{char}(\mathbb{K}) = p \neq \text{char}(\mathbb{K}_v)$ "mixed characteristic" $p$
- $\text{char}(\mathbb{K}) = \text{char}(\mathbb{K}_v) = p$ "positive characteristic/p $p$" $p$

Remark. One can retrieve the valuation up to an isomorphism of ordered abelian groups.

$$v \in \mathcal{O}_v \quad \xrightarrow{\phi_{\mathcal{O}_v}} \quad e \quad \xrightarrow{\sigma} \quad G_{\mathcal{O}_v}$$

$\text{ord}_{\mathcal{O}_v}$

Have $1:1$ correspondence:

$\{\text{valuations on } \mathbb{K}/\mathcal{O}_v\} \leftrightarrow \{\text{valuation rings on } \mathbb{K}\}$

Def. We say $v$ is definable if $\mathcal{O}_v, v$ is $\mathcal{L}(\mathbb{K})$-definable.

Examples:
- $(\mathbb{Q}, v_p)$: $\mathcal{O}_p = \mathbb{Z}_p$ = $\left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \text{p does not divide } b \right\} \subseteq \mathbb{Q}$
- $m_v = \mathbb{Z}_p$
- $\mathbb{Q}_p = \mathbb{Z}_p/\mathbb{Z}_p = \mathbb{Z}_p - F_p$
- $(\mathbb{Q}_p, v_p)$: $\mathcal{O}_p = \mathbb{Z}_p$, the p-adic integers, $\mathcal{O}_p v_p = F_p$
- $(\mathbb{K}, v_{\text{triv}})$: $\mathcal{O}_v = \mathbb{K}$, $m_{v_{\text{triv}}} = \{0\}$, $K_{v_{\text{triv}}} = K/\mathcal{O}_v = K$

Def. $(\mathbb{K}, v)$ valued field is called Henselian if there is a unique valuation $v$ on $\mathbb{K}/\mathcal{O}_v$ extending $v$.

Examples:
- $(\mathbb{Q}, v_p)$ not Henselian, $(\mathbb{Q}_p, v_p)$ Henselian
- $(\mathbb{K}, v_{\text{triv}})$ always Henselian.

A field $\mathbb{K}$ is called Henselian if there is at least one non-trivial Henselian valuation on $\mathbb{K}$.
Section (4) Given a field $K$, when is there a definable non-trivial henselian valuation on $K$?

→ $K$ should be henselian

→ $K$ should not be separably closed
  (separably closed fields are stable, and in stable structure
  no non-trivial valuation is definable)

§ The canonical henselian valuation

Recall the value group can be higher rank ($\neq R$)
  → can have proper non-trivial convex subgroups $\Delta \subseteq \Gamma$

Example $\Gamma = R$: $10$3 and $\Gamma$ are the only convex subgroups $\Gamma \subseteq \mathbb{Z}$

$\Gamma = \mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $10 \mathbb{Z}$

$\Delta = \mathbb{Z} \times \mathbb{Z}$, $10 \mathbb{Z}$

 correspondence: $\{\text{convex subgroups of } \mathbb{V}_K\} \leftrightarrow \{\text{rings}\}$
  (rings are always valuation rings)

Def $H(K) = \{\sigma, \epsilon \in K: \sigma \text{ henselian} \}$

→ $H_1(K) = \{\sigma, \epsilon \in H(K): \epsilon$ not separably closed $\}$

→ $H_2(K) = \{\sigma, \epsilon \in H(K): \epsilon$ separably closed $\}$

$H(K)$ is partially ordered by the covering relation/inclusion

Fact: the valuations in $H_2$ are coarsely ordered and coarser than (coverings) of the valuations in $H_1$

Picture:

Def the canonical henselian valuation on $K$, $\mathbb{V}_K$, is
  → the finest valuation in $H_1$, if $H_1 \neq \emptyset$
  → the coarsest valuation in $H_2$, otherwise

The answer to (4) depends on properties of $\mathbb{V}_K$

P1. $\text{char}(K) > 0$ and only need $\Gamma = \emptyset$

Theorem (Jahnke-Keininger, 2017; K-Ramello-Seceley, 2024)

$K$ henselian, not separably closed

If $\text{char}(K) = 0$, assume $K$ is perfect

If $\text{char}(K) = p > 0$, $\text{char}(K) = p > 0$, then assume $\mathbb{V}_K$ is semi-perfect.

Then,

→ $K$ admits a non-trivial definable henselian valuation

1. $K$ not henselian

2. $\text{char}(K)$ not definable

3. $\mathbb{V}_K$ not definable

4. $\text{char}(K)$ not definable

5. $\mathbb{V}_K$ not definable

6. $\text{char}(K)$ not definable

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} \hline \text{property} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \text{valuation} & \text{yes} & \text{yes} & \text{yes} & \text{yes} & \text{yes} & \text{yes} \\ \hline \text{existence} & \text{no} & \text{no} & \text{no} & \text{no} & \text{no} & \text{no} \\ \hline \end{array} \]
§ Defect

Fact (fundamental inequality)
(K\!,v) \text{- henselian, L/k finite extension } \implies \text{unique extension v to L}

\[ [L:K] = (vL: vK)[L:vK] \]  

(XX)

Def \ ((K\!,v) \in (L,v)) \text{ is defect} \iff \text{ (XX) is an equality}

(K\!,v) \text{ defect} \iff \text{all its finite extensions are}

Otherwise: Defect

Example \ \text{char(K) = 0 } \implies \text{ (K\!,v) defectless}

From now on, assume \ \text{char(K) > 0}

Def (Hochster-Parshin) \ Let (K\!,v) \subseteq (L,v) \text{ be a Galois defect extension of degree p. Let Gal(L/K) = \langle \sigma \rangle}

\[ \Sigma_L = \{ \sigma^i \in \text{Gal}(L/K) \mid \sigma^i \in L^v \} \subseteq vL = vK\]

We say \ ((K\!,v) \in (L,v)) \text{ has independent defect if there is an element H \subseteq_{\text{finite}} vK \text{ s.t.}

- vK/H has no smallest positive element
- \Sigma_L = \{ \alpha \in vK \mid \alpha > 0 \}

Proof Idea \ (\text{of } (5) \implies \text{ a non-trivial defect, val})

Assume \ (K\!,v) \text{ has defect } (5)

Find a Galois defect extension of degree p:

\[ K \subseteq K' \subseteq L \]

Perfect/Semi-Perfect \implies \text{ has independent defect}

\[ \Sigma_L \text{ & the coarsening of } \sigma \text{ corresponding to } H \]

are "essentially" \text{-definable} (Beth's definability theorem)

yields a non-trivial definable henselian valuation on K.