Cut-Elimination, Cut-Restriction

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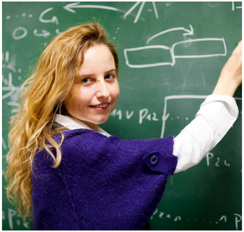
Revised Gentzen method, v2:

1. Find a sequent calculus for the logic.
2. Prove ~~cut-elimination~~ cut-restriction.

↑ This talk!



Agata Ciabattoni (TU Vienna)

Revantha Ramanayake (RU Groningen)

Introduction

The Sequent Calculus and Cut-Elimination

Exercise: Show $\vdash_{IL} (A \to B) \land (B \to C) \to (A \to C)$.

Exercise: Show $\vdash_{IL} (A \to B) \wedge (B \to C) \to (A \to C)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,{}^{1} \qquad \dfrac{\dfrac{}{(A \to B) \wedge (B \to C)}\,{}^{2}}{A \to B}}{B} \qquad \dfrac{\dfrac{}{(A \to B) \wedge (B \to C)}\,{}^{2}}{B \to C}}{C}}{A \to C}\,{}^{1}}{(A \to B) \wedge (B \to C) \to (A \to C)}\,{}^{2}$$

Exercise: Show $\nvdash_{IL} p \vee \neg p$

Exercise: Show $\nvdash_{IL} p \vee \neg p$

$$\frac{\stackrel{?}{\vdots}\atop p}{p \vee \neg p} \qquad \frac{\stackrel{?}{\vdots}\atop \neg p}{p \vee \neg p} \qquad \frac{\stackrel{?}{\vdots}\atop A \quad \stackrel{?}{\vdots}\atop A \to (p \vee \neg p)}{p \vee \neg p}$$

I tried very hard but couldn't find a proof… ???

The space of all well-formed proofs is incomprehensibly large.

$$\dfrac{\overset{?}{\vdots}\atop A \qquad \overset{?}{\vdots}\atop A \to (p \vee \neg p)}{p \vee \neg p}$$

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$$\dfrac{\overset{?}{\vdots}\;\; \\ A \qquad \overset{?}{\vdots}\; A \to (p \vee \neg p)}{p \vee \neg p}$$

$A =$

$$p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg(p\wedge(r\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to$$
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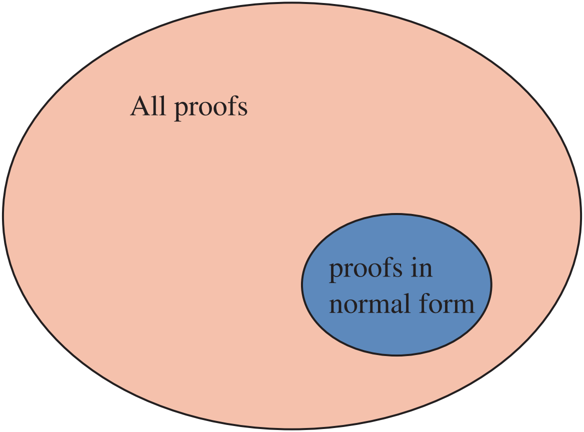
Solution: Normal forms



Gerhard Gentzen (1909-1945)

"Investigations into logical deduction" (1935)

SQ (sequent calculus) —cut elimination→ cutfree **SQ**

↑ ↓

ND (natural deduction) - - - → normalized **ND**

Sequent Calculus

SQ $\equiv$ a meta-calculus for constructing **ND** proofs.

$$\underbrace{A_1, \ldots, A_n \Rightarrow B}_{\text{sequent}}$$

"there exists a proof of B from assumptions $A_1, \ldots, A_n$"

LJ

$$\frac{}{A \Rightarrow A}\ (id)$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi}\ (e_l) \quad \frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (c_l) \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w_l) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (w_r)$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (cut)$$

$$\frac{}{\bot \Rightarrow \Pi}\ (\bot)$$

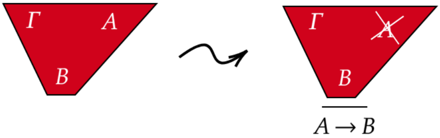
$$\frac{\Gamma, A \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi} \quad \frac{\Gamma, B \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi}\ (\wedge_l) \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\wedge_r)$$

$$\frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \Rightarrow \Pi}\ (\vee_l) \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ (\vee_r)$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow \Pi}{\Gamma, \Delta, A \rightarrow B \Rightarrow \Pi}\ (\rightarrow_l) \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ (\rightarrow_r)$$

rule in **SQ** $\equiv$ proof transformation in **ND**.

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to_R)$$



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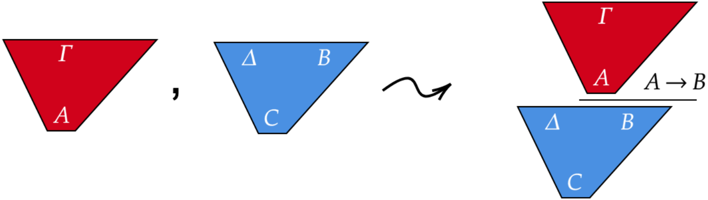

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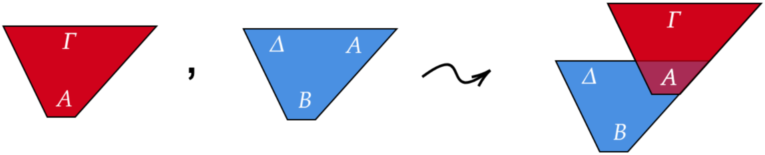

proof in **SQ** $\equiv$ recipe for constructing an **ND**-proof.

sequent calculus:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee B} \quad \dfrac{C \Rightarrow C}{C \wedge D \Rightarrow C}}{(A \vee B) \to (C \wedge D), A \Rightarrow C}}{(A \vee B) \to (C \wedge D) \Rightarrow A \to C}$$

natural deduction:

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natural deduction:

$$A$$

$$C$$

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$$\dfrac{\dfrac{\dfrac{A}{A \vee B}}{C \wedge D}}{C}$$

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natural deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]}{A \vee B} \qquad A \vee B \to C \wedge D}{C \wedge D}}{C}}{A \to C}$$

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- They have the subformula property:
  Only subformulas of the proven theorem appear

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Proofs constructed without cut are particularly nice.

- They have the subformula property:
  Only subformulas of the proven theorem appear
- They have no detours, e.g.

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}}{\begin{matrix}A \\ \vdots\end{matrix}}$$

Theorem (Gentzen 1934)

Every proof in $\mathbf{LJ}$ *(and* $\mathbf{LK}, \mathbf{LJ}^{\forall}, \mathbf{LK}^{\forall}$*) can be rewritten into an equivalent proof that does not use cuts.*

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Corollaries:

1. consistency of **CL** and **IL**
2. decidability of propositional **IL**
3. midsequent theorem for prenex formulas
4. consistency of arithmetic (w/o induction 1934, complete 1936)

$$\frac{\overset{?}{A} \quad A \to (\overset{?}{p} \vee \neg p)}{p \vee \neg p}$$

$$\begin{aligned} A = {} & p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg(p\wedge(r\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to \\ & \neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg(p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg p)))))))\to(q\vee(\neg q\to \\ & \neg(\neg p\to(q\vee\neg r)))))))\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to\neg(\neg(p\wedge((p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg(p\wedge(r\to \\ & \neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg(p\wedge(q\to\neg(\neg(p\wedge(q\to \\ & \neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg p)))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee\neg r)))))))\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to \\ & \neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg p)))))))\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg r)))))))))))))\to(q\vee(\neg q\to \\ & \neg(\neg p\to(q\vee\neg r)))))))))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee\neg r)))))))\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to\neg(\neg(p\wedge(q\to \\ & \neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg p)))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee\neg r)))))))))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee \\ & \neg r)))))))\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg p)))))))\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg r)))))))))))))\to(q\vee(\neg q\to \\ & \neg(\neg p\to(q\vee\neg r)))))))))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee\neg r)))))))\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to \\ & (q\vee(\neg r\to\neg(\neg p\to(q\vee\neg p)))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee\neg r)))))))))))))\to(q\vee(\neg q\to\neg(\neg p\to(q\vee\neg r)))))) \end{aligned}$$

$$
\dfrac{\overset{?}{\vdots}\;A \qquad A \to (\overset{?}{\vdots}\,p \vee \neg p)}{p \vee \neg p}
$$

$$
\begin{aligned}
A = \;& p\wedge(q\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg(p\wedge(r\to\neg(\neg(p\wedge(q\to\neg(\neg p\to(q\vee(\neg r\to\neg(\neg p\to(q\vee\neg(p\wedge(q\to \\
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\end{aligned}
$$

Not normal, as $A \notin \mathrm{subf}(p \vee \neg p)$.

Revised Gentzen method, v1

Generalising the Sequent Calculus

- Gentzen's method

  "find a sequent calculus, prove cut-elimination"

  works for many other logics

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  - many intermediate logics $L = \mathbf{IL} + A$,
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  - Bi-intuitionistic logic **BiInt**

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A (formula) $\rightsquigarrow$ $\Gamma \Rightarrow A$ (sequent (1935)) $\rightsquigarrow$ $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$ (hypersequent (1971))

$\rightsquigarrow$ $[\Gamma_1, [\Gamma_2 \Rightarrow \Gamma_3] \Rightarrow A_1] \Rightarrow A_2$ (nested sequent (1992?)) $\rightsquigarrow \ldots$

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- **S5** and **G** have a hypersequent system with cut elimination
- **BiInt** has a nested sequent system with cut elimination
- ...

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Caveat: The more complex the calculus, the less useful is cut-elimination!

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \alpha, \Delta \Rightarrow \Pi}{G \mid \Gamma, \Delta \Rightarrow \Pi}\ (cut) \qquad \frac{}{G \mid \alpha \Rightarrow \alpha}\ (init) \qquad \frac{G \mid \Gamma \Rightarrow \Pi}{G \mid 1, \Gamma \Rightarrow \Pi}\ (1l) \qquad \frac{}{G \mid\ \Rightarrow 1}\ (1r)$$

$$\frac{G \mid \alpha, \beta, \Gamma \Rightarrow \Pi}{G \mid \alpha \cdot \beta, \Gamma \Rightarrow \Pi}\ (\cdot\, l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Delta \Rightarrow \beta}{G \mid \Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot\, r) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow 0}\ (0r) \qquad \frac{}{G \mid 0 \Rightarrow}\ (0l)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \beta, \Delta \Rightarrow \Pi}{G \mid \Gamma, \alpha \rightarrow \beta, \Delta \Rightarrow \Pi}\ (\rightarrow l) \qquad \frac{G \mid \alpha, \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \rightarrow \beta}\ (\rightarrow r) \qquad \frac{G \mid \alpha_i, \Gamma \Rightarrow \Pi}{G \mid \alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{}{G \mid \bot, \Gamma \Rightarrow \Pi}\ (\bot l)$$

$$\frac{G \mid \alpha, \Gamma \Rightarrow \Pi \quad G \mid \beta, \Gamma \Rightarrow \Pi}{G \mid \alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{G \mid \Gamma \Rightarrow \alpha_i}{G \mid \Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r) \qquad \frac{}{G \mid \Gamma \Rightarrow \top}\ (\top r)$$

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (EW) \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ (EC)$$

The inference rules of **FLe** are obtained by dropping '$G \mid$' and removing (EW), (EC).

Figure 1. Inference Rules of HFLe

Ciabattoni/Galatos/Terui: From axioms to analytic rules in nonclassical logics

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, x : \top \vdash_G \Delta}\ \top L \qquad \frac{}{\Gamma \vdash_G x : \top, \Delta}\ \top R$$

$$\frac{\Gamma, x : A, x : B \vdash_G \Delta}{\Gamma, x : A \wedge B \vdash_G \Delta}\ \wedge L \qquad \frac{\Gamma \vdash_G x : A, \Delta \quad \Gamma \vdash_G x : B, \Delta}{\Gamma \vdash_G x : A \wedge B, \Delta}\ \wedge R$$

$$\frac{}{\Gamma, x : \bot \vdash_G \Delta}\ \bot L \qquad \frac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G x : \bot, \Delta}\ \bot R$$

$$\frac{\Gamma, x : A \vdash_G \Delta \quad \Gamma, x : B \vdash_G \Delta}{\Gamma, x : A \vee B \vdash_G \Delta}\ \vee L \qquad \frac{\Gamma \vdash_G x : A, x : B, \Delta}{\Gamma \vdash_G x : A \vee B, \Delta}\ \vee R$$

$$\frac{\Gamma, x : A \supset B \vdash_G x : A, \Delta \quad \Gamma, x : B \vdash_G \Delta}{\Gamma, x : A \supset B \vdash_G \Delta}\ \supset L \qquad \frac{\Gamma, y : A \vdash_{G \oplus_x (x,y)} y : B, \Delta}{\Gamma \vdash_G x : A \supset B, \Delta}\ \supset R$$

$$\frac{\Gamma, y : A \vdash_{(y,x) \oplus_x G} y : B, \Delta}{\Gamma, x : A \prec B \vdash_G \Delta}\ \prec L \qquad \frac{\Gamma \vdash_G x : A, \Delta \quad \Gamma, x : B \vdash_G x : A \prec B, \Delta}{\Gamma \vdash_G x : A \prec B, \Delta}\ \prec R$$

$$\frac{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, A, B \Rightarrow \Delta}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, A \wedge B \Rightarrow \Delta} \wedge_L \qquad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, B}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, A \wedge B} \wedge_R$$

$$\frac{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, A \Rightarrow \Delta \quad \Rightarrow \mid\mid \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, A \vee B \Rightarrow \Delta} \vee_L \qquad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, A, B}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B} \vee_R$$

$$\frac{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \Rightarrow \mid\mid \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow_L \qquad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, A \Rightarrow \Delta, B}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow_R$$

The crown initial structures and the crown modal rules:

$$\frac{}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, p \Rightarrow \Delta, p} \mathsf{Init} \quad \frac{}{\Rightarrow \mid\mid \mathcal{H} \mid \Gamma, \bot \Rightarrow \Delta} \bot_L \quad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma, A \Rightarrow \Pi}{\Rightarrow \mid\mid \mathcal{H} \mid \Box A \Rightarrow \mid \Sigma \Rightarrow \Pi} \mathsf{5} \quad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Rightarrow A}{\Rightarrow \mid\mid \mathcal{H} \mid \Rightarrow \Box A} \mathsf{K}$$

The crown structural rules:

$$\frac{\Rightarrow \mid\mid \mathcal{H} \mid \Omega \Rightarrow \Xi \mid \Omega \Rightarrow \Xi}{\Rightarrow \mid\mid \mathcal{H} \mid \Omega \Rightarrow \Xi} \mathsf{EC} \qquad \frac{\Rightarrow \mid\mid \mathcal{H}}{\Rightarrow \mid\mid \mathcal{H} \mid \Omega \Rightarrow \Xi} \mathsf{EW}$$

$$\frac{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma, A, A \Rightarrow \Pi}{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma, A \Rightarrow \Pi} \mathsf{IC}_L \qquad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma \Rightarrow \Pi, A, A}{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma \Rightarrow \Pi, A} \mathsf{IC}_R \qquad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma \Rightarrow \Pi}{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma, \Omega \Rightarrow \Pi, \Xi} \mathsf{IW}$$

Fig. 2. The crown rules of the calculus for **K5**

$$\mathsf{id_P}\,\frac{}{\mathcal{G}\{p,\overline{p}\}} \qquad \mathsf{id}_\top\,\frac{}{\mathcal{G}\{\top\}} \qquad \wedge\,\frac{\mathcal{G}\{\varphi\wedge\psi,\varphi\} \quad \mathcal{G}\{\varphi\wedge\psi,\psi\}}{\mathcal{G}\{\varphi\wedge\psi\}}$$

$$\vee\,\frac{\mathcal{G}\{\varphi\vee\psi,\varphi,\psi\}}{\mathcal{G}\{\varphi\vee\psi\}} \qquad \Box_t\,\frac{\mathcal{G},\Box\varphi,[\varphi]}{\mathcal{G},\Box\varphi} \qquad \Box_{t'}\,\frac{[\Sigma,\Box\varphi],[\varphi]}{\Sigma,\Box\varphi} \qquad \Diamond_t\,\frac{\mathcal{G},\Diamond\varphi,[\Sigma,\varphi]}{\mathcal{G},\Diamond\varphi,[\Sigma]}$$

$$\Box_c\,\frac{\mathcal{G},[\![\Sigma,\Box\varphi]\!],[\varphi]}{\mathcal{G},[\![\Sigma,\Box\varphi]\!]} \qquad \Box_{c'}\,\frac{\mathcal{G},[\![\Sigma,\Box\varphi]\!],[[\varphi]]}{\mathcal{G},[\![\Sigma,\Box\varphi]\!]} \qquad \Diamond_c\,\frac{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!],(\!|\Pi,\varphi|\!)}{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!],(\!|\Pi|\!)}$$

$$\mathsf{d}_t\,\frac{\mathcal{G},\Diamond\varphi,[\varphi]}{\mathcal{G},\Diamond\varphi} \qquad \mathsf{d}_c\,\frac{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!],[\varphi]}{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!]} \qquad \mathsf{d}_{c'}\,\frac{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!],[[\varphi]]}{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!]} \qquad \mathsf{t}\,\frac{\mathcal{G},[\![\Sigma,\Diamond\varphi,\varphi]\!]}{\mathcal{G},[\![\Sigma,\Diamond\varphi]\!]}$$

Revised Gentzen Method, v2

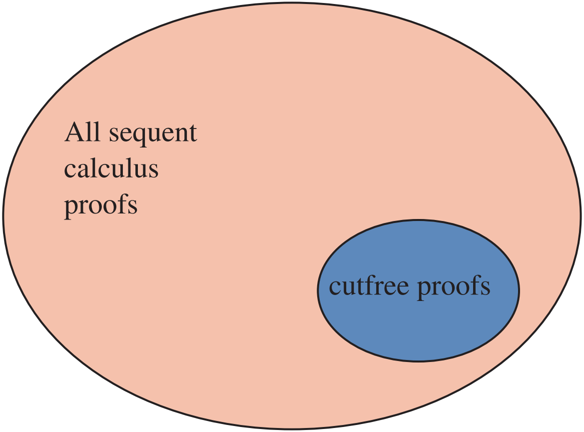
Cut-restriction

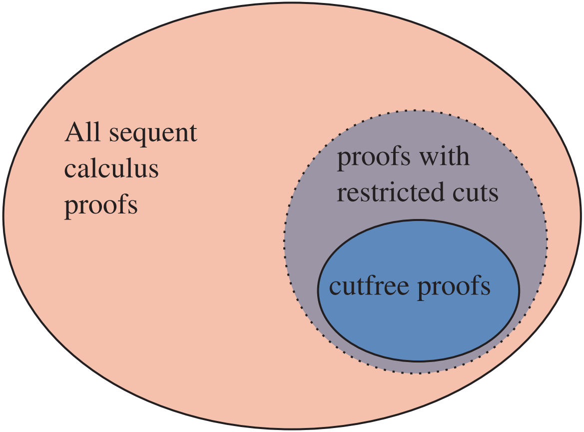
Revised Gentzen method, v1:

1. Find a ~~sequent calculus~~ some calculus for the logic.
2. Prove cut-elimination.

Revised Gentzen method, v2:

1. Find a sequent calculus for the logic.
2. Prove ~~cut-elimination~~ cut-restriction.


All sequent
calculus
proofs
cutfree proofs


All sequent
calculus
proofs
proofs with
restricted cuts
cutfree proofs

A particular cut-restriction

A sequent calculus has the analytic cut property (ACP) if every provable sequent has a proof using only analytic cuts:

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{cut} \qquad A \in \mathrm{subf}(\Gamma, \Delta)$$

A particular cut-restriction

A sequent calculus has the analytic cut property (ACP) if every provable sequent has a proof using only analytic cuts:

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ cut \qquad A \in \mathrm{subf}(\Gamma, \Delta)$$

Theorem (Kowalski/Ono 2017)

Let S be a sequent calculus with cut as the only non-analytic rule. TFAE:

1. S has the global subformula property
2. S has the local subformula property
3. S has the analytic cut property

Two cut-restriction results from the literature

Theorem (Takano 1992)

S5 has a sequent calculus with the ACP.

Theorem (Kowalski/Ono 2017)

BiInt has a sequent calculus with the ACP.

A cut-restriction result for intermediate logics

Theorem (Ciabattoni, Ramanayake, L. 2021)

If $\mathbf{IL} + A$ *has a cutfree hypersequent calculus, then* $\mathbf{IL} + A$ *has a sequent calculus where only* *<u>set-restricted axiom cuts</u>* *are needed.*

$A = A(p,q)$

$$\dfrac{\Gamma, A(\bigwedge_i \psi_i, \bigwedge_j \rho_j) \Rightarrow B}{\Gamma \Rightarrow B}\ (cut)$$
$$\vdots$$
$$\Rightarrow \varphi$$

where $\psi_i, \rho_j \in \mathrm{subf}(\varphi)$.

(Remark: can be generalised to substructural logics)

calculus-free reformulation

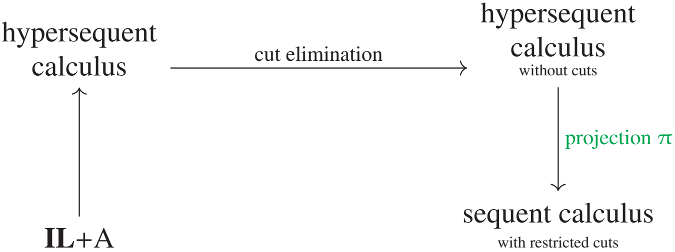
Theorem (Ciabattoni, Ramanayake, L. 2021)

If $\mathbf{IL} + A$ *has a cutfree hypersequent calculus, then* $\mathbf{IL} + A$ *satisfies a* *refined deduction theorem:*

$$\varphi \in \mathbf{IL} + A \iff \left((\bigwedge_{i=1}^{n} A_i) \to \varphi \right) \in \mathbf{IL}$$

for some set-restricted instances $A_1, \ldots, A_n$ *of* A.

proof via projections



π only introduces set-restricted cuts.

The π-transformation (1/3)

Assume $F \in \mathbf{LJ} + (p \to q) \vee (q \to p)$.

$$\dfrac{\overset{\delta}{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \quad \overset{\gamma}{G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}}{G \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ com$$

$$\vdots$$

$$\Rightarrow F$$

LEFT SPLIT

$$\dfrac{\dfrac{\Delta_2 \Rightarrow \bigwedge\Delta_2 \quad \overset{\delta}{G \mid \Gamma_1, \bigwedge\Delta_1 \Rightarrow \Pi_1}}{G \mid \boxed{\bigwedge\Delta_2 \to \bigwedge\Delta_1}, \Gamma_1, \Delta_2 \Rightarrow \Pi_1}\ \to_L}{G \mid \boxed{\bigwedge\Delta_2 \to \bigwedge\Delta_1}, \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ ew$$

$$\vdots$$

$$\boxed{\bigwedge\Delta_2 \to \bigwedge\Delta_1} \Rightarrow F$$

The π-transformation (2/3)

$$\dfrac{\overset{\delta}{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \qquad \overset{\gamma}{G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}}{G \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ com$$

$$\vdots$$

$$\Rightarrow F$$

RIGHT SPLIT

$$\dfrac{\dfrac{\Delta_1 \Rightarrow \bigwedge\Delta_1 \qquad \overset{\gamma}{G \mid \Gamma_2, \bigwedge\Delta_2 \Rightarrow \Pi_2}}{G \mid \boxed{\bigwedge\Delta_1 \to \bigwedge\Delta_2}, \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ \to_L}{G \mid \boxed{\bigwedge\Delta_1 \to \bigwedge\Delta_2}, \Gamma_2, \Delta_1 \Rightarrow \Pi_2 \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1}\ ew$$

$$\vdots$$

$$\boxed{\bigwedge\Delta_1 \to \bigwedge\Delta_2} \Rightarrow F$$

The π-transformation (3/3)

$$
\dfrac{
\dfrac{
\begin{array}{c}\text{LEFT SPLIT}\\ \vdots \\ \boxed{\bigwedge\Delta_2 \to \bigwedge\Delta_1} \Rightarrow F\end{array}
\qquad
\begin{array}{c}\text{RIGHT SPLIT}\\ \vdots \\ \boxed{\bigwedge\Delta_1 \to \bigwedge\Delta_2} \Rightarrow F\end{array}
}{
\underbrace{\boxed{(\bigwedge\Delta_2 \to \bigwedge\Delta_1) \vee (\bigwedge\Delta_1 \to \bigwedge\Delta_2)} \Rightarrow F}_{:=A}
}\ \vee_L
}{
\Rightarrow F
}\ \mathit{cut}
$$

$$\Delta_1, \Delta_2 \subseteq \mathrm{Subf}(F)$$

Recall:

Theorem (Takano 1992)

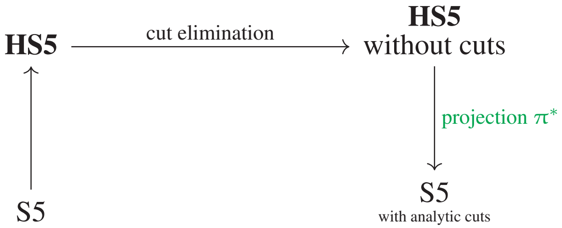
The sequent calculus for **S5** is complete if one admits analytic cuts.

Theorem (Kowalski/Ono 2017)

The sequent calculus for **BiInt** is complete if one admits analytic cuts.

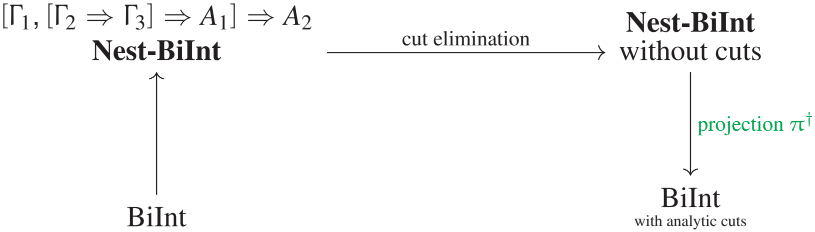
analytic $>$ set-restriced

Also C/R/L 2021 (reproving Takano ’92)



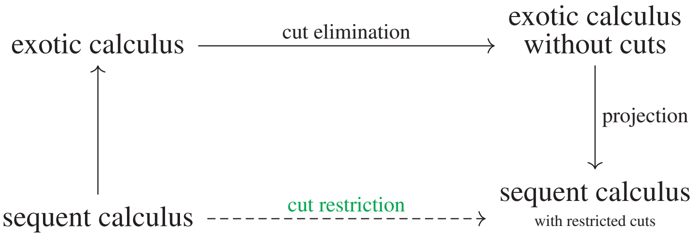
π^* only introduces analytic cuts.

unpublished 2022 (first syntactic proof, reproving Kowalski/Ono 2017)



$\pi^\dagger$ only introduces analytic cuts.

The general scheme



Q: *Can we do without the detour through exotic calculi?*

sequent calculus - - - cut restriction - - -> sequent calculus with analytic cuts

Theorem (Ciabattoni/L./Ramanayake 2023)

There is a sound and terminating algorithm that eliminates non-analytic cuts for all sequent calculi satisfying ♥.

Therefore all calculi satisfying ♥ have the ACP.

sequent calculus - - - - cut restriction - - - -> sequent calculus with analytic cuts

Theorem (Ciabattoni/L./Ramanayake 2023)

There is a sound and terminating algorithm that eliminates <u>non-analytic cuts</u> for all sequent calculi satisfying ♥.

Therefore all calculi satisfying ♥ have the ACP.

S5 ✓ **BiInt** ✓ multi-modal **S5** ✓ **G4** ✓ $\mathbf{BiInt}^{\mathbf{S5}}$ ✓

The algorithm

All standard cut-reduction steps +

$$
\dfrac{
\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Rightarrow C_1, E\end{matrix} \qquad \begin{matrix}\vdots\ \delta_2 \\ C_2 \Rightarrow E\end{matrix}}{\Rightarrow C_1 \prec C_2, E}\ (\prec_R)
\qquad
\dfrac{\begin{matrix}\vdots\ \pi_2 \\ C_1 \prec C_2, D_1 \Rightarrow D_2\end{matrix}}{C_1 \prec C_2 \Rightarrow D_1 \to D_2}\ (\to_R)
}{\Rightarrow D_1 \to D_2, E}\ (cut)
\qquad\qquad
\dfrac{\begin{matrix}\vdots\ \pi_1 \\ C_1 \Rightarrow C_2, F\end{matrix}}{C_1 \prec C_2 \Rightarrow F}\ (\prec_L)
$$

$$\Downarrow$$

$$
\dfrac{
\dfrac{
\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Rightarrow C_1, E\end{matrix} \qquad \begin{matrix}\vdots\ \pi_1 \\ C_1 \Rightarrow C_2, F\end{matrix}}{\Rightarrow C_2, E, F}\ (cut)''
\qquad
\begin{matrix}\vdots\ \delta_2 \\ C_2 \Rightarrow E\end{matrix}
}{\Rightarrow E, F}\ (cut)'''
\qquad
\dfrac{\begin{matrix}\dfrac{}{F \Rightarrow F} \\ \vdots\ \pi_2[F / C_1 \prec C_2] \\ F, D_1 \Rightarrow D_2\end{matrix}}{F \Rightarrow D_1 \to D_2}\ (\to_R)
}{\Rightarrow D_1 \to D_2, E}\ (cut')
$$

The algorithm

All standard cut-reduction steps +

$$\dfrac{\begin{matrix}\vdots\ \pi_1\\ C_1 \Rightarrow C_2, {\color{green}F}\end{matrix}}{{\color{red}C_1 \prec C_2} \Rightarrow {\color{green}F}}\ (\prec_L)$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Rightarrow C_1, E\end{matrix} \qquad \begin{matrix}\vdots\ \delta_2\\ C_2 \Rightarrow E\end{matrix}}{\Rightarrow {\color{red}C_1 \prec C_2}, E}\ (\prec_R) \qquad \dfrac{\begin{matrix}\vdots\ \pi_2\\ {\color{red}C_1 \prec C_2}, D_1 \Rightarrow D_2\end{matrix}}{{\color{red}C_1 \prec C_2} \Rightarrow D_1 \to D_2}\ (\to_R)}{\Rightarrow D_1 \to D_2, E}\ (cut)$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Rightarrow C_1, E\end{matrix} \qquad \begin{matrix}\vdots\ \pi_1\\ C_1 \Rightarrow C_2, {\color{green}F}\end{matrix}}{\Rightarrow C_2, E, {\color{green}F}}\ (cut)'' \qquad \begin{matrix}\vdots\ \delta_2\\ C_2 \Rightarrow E\end{matrix}}{\Rightarrow E, {\color{green}F}}\ (cut)''' \qquad \dfrac{\begin{matrix}\overline{{\color{green}F} \Rightarrow {\color{green}F}}\\ \vdots\ \pi_2[{\color{green}F}/{\color{red}C_1 \prec C_2}]\\ {\color{green}F}, D_1 \Rightarrow D_2\end{matrix}}{{\color{green}F} \Rightarrow D_1 \to D_2}\ (\to_R)}{\Rightarrow D_1 \to D_2, E}\ (cut')$$

♥ = the substitution is well-defined

A strengthening for **S5**

The only necessary cut formulas are

1. subformulas of the lower sequent of cut
2. are prefixed with $\Box$, and
3. appear in the scope of another $\Box$.

$$A(\ldots \Box(\ldots \Box C) \ldots)$$

A strengthening for **S5**

The only necessary cut formulas are

1. subformulas of the lower sequent of cut
2. are prefixed with $\Box$, and
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$$A(\ldots\Box(\ldots\Box C)\ldots)$$

$$\dfrac{\dfrac{\dfrac{\Box A \Rightarrow \Box A}{\Rightarrow \Box A, \neg\Box A}\,(\neg_R)}{\Rightarrow \Box A, \Box\neg\Box A}\,(5) \qquad \dfrac{A \Rightarrow A}{\Box A \Rightarrow A}\,(T)}{\Rightarrow \Box\neg\Box A, A}\,(cut)$$

Corollary: No cuts needed for modal depth $\leqslant 1$.

Conclusion

Back to the bigger picture!

Revised Gentzen method, v1:

1. Find a ~~sequent calculus~~ some calculus for the logic.
2. Prove cut-elimination.

Revised Gentzen method, v2:

1. Find a sequent calculus for the logic.
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- the standard approach
- tons of results

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- not so well understood
- some scattered results, some recent progress

$\Rightarrow$ lots to do!

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- not so well understood
- some scattered results, some recent progress

$\Rightarrow$ lots to do!

Our TODO list

- direct proof of set-boundedness for $\mathbf{IL} + A$
- analytic cuts for K-type rules (relax ♥)
  UPDATE: Ciabattoni/Tesi IJCAR 2024
- properties weaker than ACP, e.g. for **K5**
- cyclic proofs
- ...

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How are these methods and their results related?

Are they two sides of the same coin?

Or is one of them better?

S5 (Modal logic of equivalence relations)

Ohnishi/Matsumoto 1957

$$
\dfrac{\dfrac{\dfrac{\Box A \Rightarrow \Box A}{\Rightarrow \Box A, \neg\Box A}\,(\neg_R)}{\Rightarrow \Box A, \Box\neg\Box A}\,(5) \qquad \dfrac{A \Rightarrow A}{\Box A \Rightarrow A}\,(T)}{\Rightarrow \Box\neg\Box A, A}\,(cut)
$$

BiInt (Bi-Intuitionistic logic)

Rauszer 1975

$$\dfrac{\dfrac{A \prec B, C \Rightarrow A \prec B}{A \prec B \Rightarrow C \to (A \prec B)}\,(\to_L) \qquad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow A \prec B, B}\,(\prec_R)}{A \Rightarrow B, C \to (A \prec B)}\,(cut)$$

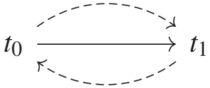
example due to Pinto/Uustalu 2003

Truth tables

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$$A \vee \Box_F \Diamond_P \neg A$$

1. Assume $t_0 \nvDash A$ and $t_0 \nvDash \Box_F \Diamond_P \neg A$.
2. Hence $\exists t_1 \geqslant t_0$ such that $t_1 \nvDash \Diamond_P \neg A$.
3. Hence $\forall s \leqslant t_1$, $s \vDash A$.
4. Therefore $t_0 \models A$. $\lightning$ (1)



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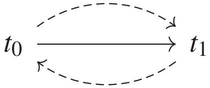
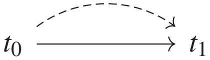


Not formalizable in an analytic (unlabelled) sequent calculus!

$$A \vee \Box_F \Diamond_P \neg A$$

1. Assume $t_0 \nvDash A$ and $t_0 \nvDash \Box_F \Diamond_P \neg A$.
2. Case 1: Assume $t_0 \models \Diamond_P \neg A$
   - Then $\exists t_1 \geqslant t_0$ such that $t_1 \nvDash \Diamond_P \neg A$ and $t_1 \vDash \Diamond_P \neg A$. $\lightning$
3. Case 2: Assume $t_0 \nvDash \Diamond_P \neg A$
   - Then in particular, $t_0 \vDash A$. $\lightning$ (1)

Case 1:

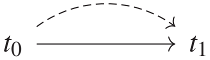


Case 2: t_0

$$A \vee \Box_F \Diamond_P \neg A$$

1. Assume $t_0 \not\models A$ and $t_0 \not\models \Box_F \Diamond_P \neg A$.
2. <u>Case 1:</u> Assume $t_0 \models \Diamond_P \neg A$
   - Then $\exists t_1 \geqslant t_0$ such that $t_1 \not\models \Diamond_P \neg A$ and $t_1 \models \Diamond_P \neg A$. $\lightning$
3. <u>Case 2:</u> Assume $t_0 \not\models \Diamond_P \neg A$
   - Then in particular, $t_0 \models A$. $\lightning$ (1)

Case 1:



Case 2: t_0

$$\dfrac{\dfrac{\Diamond_P \neg A \Rightarrow \Diamond_P \neg A}{\Diamond_P \neg A \Rightarrow \Box_F \Diamond_P \neg A}\,(\Box_F) \qquad \dfrac{\Rightarrow A, \neg A}{\Rightarrow A, \Diamond_P \neg A}\,(\Diamond_P)}{\Rightarrow A, \Box_F \Diamond_P \neg A}\,(cut)$$