de Finetti coherence and exchangeability in infinitary logic

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1. Probability in Łukasiewicz logic
2. A well behaved infinitary logic
3. A Dutch-book theorem and a description of exchangeability
4. An application to statistical models

MV-algebras

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A is an MV-algebra iff $A \in HSP([0,1])$

Mundici's equivalence

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Take (G, u) ℓu-group, u is strong order unit:

$$\text{for any } x \text{ exists } n \text{ such that } x \vee (-x) \leq nu$$

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$$([0, u]_G, \oplus, \neg, 0) \text{ is an MV-algebra}$$

Categorical equivalence. Mundici, 1986

For any MV-algebra A there exists (G, u) such that

$$A \simeq [0, u]_G$$

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if there is no way for **G** to arrange her stakes in order to win money independently of the result of the events involved in the bet.

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The total balance of this bet in the possible world v is

$$\sum_{i=1}^{n} \sigma_i \beta_i - \sum_{i=1}^{n} \sigma_i v(\varphi_i) = \sum_{i=1}^{n} \sigma_i (\beta_i - v(\varphi_i)).$$

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Replace classical logic with Łukasiewicz logic and evaluations in $\{0,1\}$ with evaluations (=homomorphisms) in $[0,1]$...

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Let A be an MV-algebra, $E = \{e_1, \ldots, e_n\} \subseteq A$ the set of events. A book $\beta \in [0, 1]^n$ on E, is said to be coherent if for any choice of stakes $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ by G, there exists $h \in Hom(A, [0, 1])$ such that

$$\sum_{i=1}^{n} \sigma_i(\beta_i - h(a_i)) \geq 0$$

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Mundici D., *Averaging the Truth-Value in Łukasiewicz Logic*, Studia Logica 55 (1995) 113-127.

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States define probabilities à la de Finetti...

The no-Dutch book theorem

Theorem

Let A be an MV-algebra, $E = \{e_1, \dots, e_n\} \subseteq A$ a finite set of events. TFAE:

1. *β is coherent.*
2. *β can be extended to a convex combination of points at most $n+1$ points in $Hom(A, [0,1])$.*
3. *there exists a state $s \in \mathcal{S}(A)$ such that $\beta_i = s(e_i)$.*

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What about infinitary logic?

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- Random variables, on the other end, have been interpreted in notions that do not have a clear role in Łukasiewicz logic. They have been called observables.
- Usually, they are homomorphisms from Borel subsets of the reals to σ-complete MV-algebras.

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We defined a logical systems that it is standard complete wrt $[0,1]$ and has, as intented semantics, RMV_σ.

Free algebras for countably-many generators in RMV_σ

Theorem (Di Nola, L., Lenzi, 2021 — Di Nola, L., Leuştean, 2018)

The free κ-generated algebra in RMV_σ is the algebra

$$Borel([0,1]^\kappa) = \{a : [0,1]^\kappa \to [0,1] \mid a \text{ is Borel-measurable}\},$$

generated by the projections π_i, $i \in \kappa$ and $\kappa \leq \omega$.

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- A Borel subset of X is an element of the σ-algebra generated by the open set of X.
- A function $a : X \to Y$ is Borel measurable if preimages of Borel sets of Y are Borel sets of X.

A privileged subclass

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V can be a also be a G_δ set: they are subsets of a topological space that are a countable intersection of open sets.

Why this setting?

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event E (or χ_E)	function $f \in \mathrm{Borel}([0,1]^\kappa)$

Conditional events and coherence

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Given conditional and unconditional events in A, a conditional book is the assignment

$$\beta\colon (p_1, q_1) \mapsto \alpha_1, \ldots (p_n, q_n) \mapsto \alpha_n, r_1 \mapsto c_1, \ldots r_m \mapsto c_m$$

where $\alpha_i, c_j \in [0, 1]$.

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A book is said complete if for any q_i there exists a unique index j such that $q_i = r_j$.

The main result

Let V be a G_δ-subset of $[0,1]^\kappa$ and and $A \simeq \mathrm{Borel}(V)$.

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is conditionally coherent if, and only if, there exists a σ-state s on A such that $s(p_i \cdot qi) = \alpha_i s(q_i)$ and $s(r_j) = c_j$, for the obvious choices of the indexes.

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Steps of the proof

1. Reduce to the case of unconditional books
2. Prove the analogous of de Finetti's theorem for MV-algebras! the same proof strategy does not work!

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But, as we deal with structure closed under countable suprema, we wanted a σ-state.

This is guaranteed by the hypothesis on V: if V is a G_δ-subset of $[0,1]^\kappa$, with κ countable, then it is a Polish space, and any linear function preserves increasing countable suprema!

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A needed ingredient for a metamathematics of statistics.

What is exchangeability

Take $(T, \mathcal{X})$ and let $\overline{\mathcal{X}}$ be the smallest σ-algebra on T^ω that contains all sets of type

$$C(E_{i_1}, \ldots, E_{i_k}) = \prod_n E_n \text{ with } E_n = T \text{ for } n \notin \{i_1, \ldots, i_k\}.$$

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A generic measure σ on $\overline{\mathcal{X}}$ is called exchangeable if for any permutation π of $\{i_1, \ldots, i_k\}$,

$$\sigma\left(C(E_{i_1}, \ldots, E_{i_k})\right) = \sigma\left(C(E_{\pi(i_1)}, \ldots, E_{\pi(i_k)})\right).$$

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A generic measure σ on $\overline{\mathcal{X}}$ is called presentable if there exists a measure ν on the set $\mathcal{P}$ of all probability measure on $\mathcal{X}$ such that for any $A \in \overline{\mathcal{X}}$,

$$\sigma(A) = \int_{\mathcal{P}} \overline{\mu}(A) d\nu(\mu).$$

where $\overline{\mu}$ is the unique product measure of μ.

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$\{f_i\}_{i\in I}$ is exchangeable if for any $k \in \mathbb{N}$ and any two sets of indexes $i_1, \ldots, i_k$, $j_1, \ldots, j_k$, the distribution laws $F_{i,k}$ of $(f_{i_1}, \ldots, f_{i_k})$ and $F_{j,k}$ of $(f_{j_1}, \ldots, f_{j_k})$ coincide.

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Distribution of a RV: for $f\colon (X, \mathcal{S}, \mu) \to (E, \mathcal{E})$, its distribution law is the probability $A \mapsto P((f \in A)) = P(\{x \in X \mid f(x) \in A\})$, that is, $A \mapsto \mu(f^{-1}(A))$

What are random variables in our setting'

Classical probability	Non-classical probability
probability measure	state
σ-algebra of events	algebra in RMV_σ
event E (or χ_E)	function $f \in \text{Borel}([0,1]^\kappa)$
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Non-classical random variables

A κ-dimensional observable on $A\mathrm{RMV}_\sigma$ is any σ-homomorphism of Riesz MV-algebras

$$\mathcal{X} : \mathrm{Borel}([0,1]^\kappa) \to A$$

Di Nola, Dvurečenskij, L., 2020, APAL

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$$f = (\mathcal{X}(\pi_i)_{i\in\kappa})$$

Why do we liked this approach?

- Random variables are measurable maps $\eta : (X, \Sigma, \mu) \to (E, \mathcal{E})$, from a probability space (X, Σ, μ) to a measurable space $(E, \mathcal{E})$.

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  Indeed,

  $$f : (X, \mathcal{S}(A), \mu_s) \to ([0,1]^\kappa, \mathcal{BA}([0,1]^\kappa)),$$

  with $\mu_s : \mathcal{S}(A) \to [0,1]$ given by $\mu_s(Y) = s(\chi_Y)$.

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Weak exchangeability, on the other hand, is manageable via states.

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For any $n \in \mathbb{N}$, let $A_n \simeq \text{Borel}(V_n)$.

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$\bigoplus_n A_n$ exists in RMV_σ and

$$\bigoplus_n A_n = \mathrm{Borel}(V) \quad \text{with } V = \prod_{n\in\mathbb{N}} V_n \subseteq [0,1]^{\bigcup_n X_n}.$$

Presentable and exchangeable states

For $A \simeq \mathrm{Borel}(V)$, a σ-state $s : \bigoplus_\omega A \to [0,1]$ is called weakly exchangeable if the associate measure on $\prod_\omega V$ is exchangeable in the classical sense.

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Theorem (Weak de Finetti's exchangeability)

Let $\kappa \leq \omega$ be a cardinal. A state on Borel$([0,1]^\kappa)$ is weakly exchangeable if, and only if, it is weakly presentable.

The case of a Boolean process of observables

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Consider $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a sequence of one-dimensional observables such that, for any $n \in \mathbb{N}$,

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Equivalently, $f_n = \chi_{E_n}$ with $E_n \in \mathcal{BO}(V)$.

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For example, a could be a step function modelling the efficiency of a tool needed to perform the experiment.

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$$s(\chi_{C_k^n}) = t(p_n(1-p_1)^{n-k}),$$

where $p_n(x) = x^n$ *and*
$C_k^n = E_{i_1} \cap \ldots \cap E_{i_k} \cap (V \setminus E_{i_{k+1}}) \cap \ldots \cap (V \setminus E_{i_n})$.

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We can use arrows in **IRL** to define logico-algebraic statistical models

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- $P \subseteq [0,1]^d$ is the set of states of the world, or parameters;
- $\eta = (\eta_i)_{i \le \kappa} \colon P \to [0,1]^k$ is our statistical model: to each parameter $\mathsf{x} \in P$ it associates the tuple $(\eta_i(\mathsf{x}))_{i \le \kappa}$.

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Coherent model

The model $\eta : P \subseteq [0,1]^d \to [0,1]^k$ is coherent with respect to the events $E = \{p_1, \ldots, p_k\} \subseteq \mathrm{Baire}([0,1]^n)$ iff for any $\mathrm{x} \in P$ there exists a state $s \colon \mathrm{Baire}([0,1]^n) \to [0,1]$ such that such that $s(p_i) = \eta_i(\mathrm{x})$ for any $i = 1, \ldots, k$

An example

Let $k \in \mathbb{N}$ and let us consider a binomial model:

$$\eta = (\eta_0, \ldots, \eta_k) \colon [0,1] \to [0,1]^{k+1},$$

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Since $\sum_{i=0}^{k} \eta_i(x) = 1$, such a model is always coherent with respect to any set $\{p_1, \dots, p_k\}$ that satisfies the following conditions:
(1) $\bigoplus_{i \neq j} p_i = \neg p_j$
(2) for any i, there exists $x \in [0,1]$ such that $p_i(x) = 1$.

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Thank you!