Constructive Fuzzy Logics

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Abstract

We propose generalizations of Kripke semantics [5] from *Intuitionistic logic* **IL** appropriate for *Intuitionistic Affine logic* **ALi**, Hajek's *Basic Logic* or **BL**, and $\mathbf{GBL}_{ewf}$. This semantics comes from our own [1] (with Oliva and Robinson) which is based in turn on the poset product construction of Jipsen and Montagna [4] and Bova and Montagna [2] for representing $\mathbf{GBL}_{ewf}$- algebras. We have proven soundness and completeness for $\mathbf{GBL}_{ewf}$ and **BL** and soundness for **ALi**, leaving completeness in that case as an open problem. We also discuss similar results for the monoidal t-norm logic or **MTL**.

Mapping the Terrain

$$
\begin{array}{ccccc}
\mathbf{ALc} & \longrightarrow & \mathbf{LLc} & \longrightarrow & \mathbf{CL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{ALi} & \longrightarrow & \mathbf{GBL}_{ewf} & \longrightarrow & \mathbf{IL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{ALm} & \longrightarrow & \mathbf{Luk}_{m} & \longrightarrow & \mathbf{ML}
\end{array}
$$

Figure: Relationships between the Logics

What makes a logic 'Substructural'?

- Remove one or more of the structural rules – contraction (Affine logics), weakening (Relevance logics), contraction and weakening (Linear Logic), Contraction and weakening and commutativity (Lambek Calculus)
- Restrictions: Just one formula on the right of the turnstyle (Intuitionistic logic, Intuitionistic Linear Logic, Lambek); restricted contraction (Lukasiewicz logic, Intermediate logics)
- But substructural logics also result from algebraic or semantic considerations, where one "unwinds" the proof theory later: e.g. Gaggles (Dunn), commuting equivalence relations (Rota), Quantales, and then **BL** . . .
- Or directly from the combinators: BCK, BCI logic

What makes a logic ‘Fuzzy’?

- Typically no contraction present.
- Often formulas evaluated in $[0, 1]$, endowed with some appropriate algebra.
- Reject excluded middle, in some cases double negation equivalences, etc.
- *Sorites Paradox* (example); cannot be expressed in classical systems because of semantics
- *Sorites* can be expressed, but is not derivable in e.g. Łukasiewicz logic
- This means the standard deduction theorem may fail for these logics.
- Hence the usual analytic proof systems are out in the fuzzy case.

Present work

- We have followed a strand of research employing Jipsen's post product construction to arrive at relational semantics for a range of fuzzy logics that are Intuitionistic, or in terms of substructural logic: extensions of Intuitionistic Affine logic.
- We first say something about $\mathbf{GBL}_{ewf}$, contrast with Intuitionistic and the famed Kripke semantics, contrast with ours, and then describe the poset products.

Intuitionistic Logic – **IL**

$$\frac{}{\Gamma, \phi \vdash \phi}\ \text{Ax}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ \to \text{I}$$

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\ \wedge \text{E} \quad (i \in \{1, 2\})$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \phi \to \psi}{\Gamma, \Delta \vdash \psi}\ \to \text{E}$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \vee \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ \wedge \text{I}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \text{ExFalso}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee \text{I} \quad (i \in \{1, 2\})$$

Figure: Intuitionistic Logic

Definition 1

A Kripke frame is $W = \langle W, \preceq \rangle$, with W a possibly empty set, and $\preceq$ a binary relation on W. Elements of W are nodes, and $\preceq$ is known as accessibility relation on W.

Definition 2

A Kripke semantics, $\mathcal{W} = \langle W, \leq, \Vdash^{\mathrm{K}} \rangle$, consists of a Kripke frame $W = \langle X, \leq \rangle$ and $\Vdash^{\mathrm{K}}$ is a relation on nodes satisfying the following conditions:

- $w \Vdash^{\mathrm{K}} p$ iff $v(p, w) = \top$
- $w \Vdash^{\mathrm{K}} \phi \wedge \psi$ iff $w \Vdash^{\mathrm{K}} \phi$ and $w \Vdash^{\mathrm{K}} \psi$
- $w \Vdash^{\mathrm{K}} \phi \vee \psi$ iff $w \Vdash^{\mathrm{K}} \phi$ or $w \Vdash^{\mathrm{K}} \psi$
- $w \Vdash^{\mathrm{K}} \phi \to \psi$ iff $\forall v : w \preceq v, ((v \Vdash^{\mathrm{K}} \phi) \to (v \Vdash^{\mathrm{K}} \psi))$
- never $w \Vdash^{\mathrm{K}} \bot$

Kripke structures and monotonicity

(M) If $v \Vdash^{\mathrm{K}} p$ and $v \preceq w$ then $w \Vdash^{\mathrm{K}} p$

($\bot$) $\neg(w \Vdash^{\mathrm{K}} \bot)$

Using the above, one obtains an essential (and well-known) property characterising the satisfaction of formulas in intuitionistic logic.

Theorem 3

The monotonicity property (M) holds for all $\mathcal{L}$-formulas ϕ, i.e.

$$\textit{if } w \Vdash^{\mathrm{K}} \phi \textit{ and } w \preceq v \textit{ then } v \Vdash^{\mathrm{K}} \phi$$

Illustration: Kripke semantics

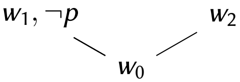

Figure: $\nvdash_{IL} p \vee \neg p$

$$
\begin{array}{c}
p \vdash_{\mathbf{IL}} p \\
\hline
p \vdash_{\mathbf{IL}} p \vee \neg p \\
\hline
\neg(p \vee \neg p), p \vdash_{\mathbf{IL}} \\
\hline
\neg(p \vee \neg p) \vdash_{\mathbf{IL}} \neg p \\
\hline
\neg(p \vee \neg p) \vdash_{\mathbf{IL}} \neg p \vee p \\
\hline
\neg(\neg p \vee p), \neg(p \vee \neg p) \vdash_{\mathbf{IL}} \\
\hline
\neg(p \vee \neg p) \vdash_{\mathbf{IL}} \\
\hline
\vdash_{\mathbf{IL}} \neg\neg(p \vee \neg p)
\end{array}
$$

Illustration: Kripke semantics

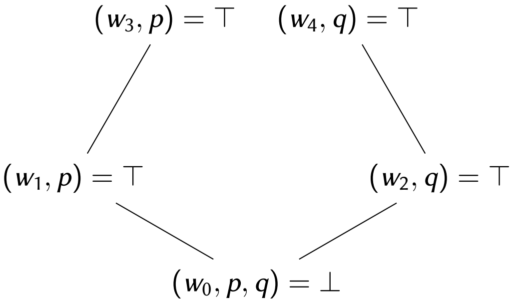

Figure: Truth grows – in a step

Illustration: Our (fuzzy) semantics

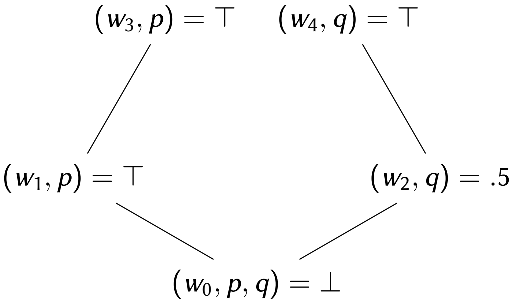

Figure: Truth grows – in a slope

$\mathbf{GBL}_{ewf}$

(A1) $\phi \to \phi$ (*identity*)

(A2) $(\phi \to \psi) \to (\psi \to \chi) \to \phi \to \chi$ (*composition*)

(A3) $\phi \otimes \psi \to \psi \otimes \phi$ (*commutativity of strong conjunction*)

(A4) $\phi \otimes \psi \to \psi$ (*projection*)

(A5) $(\phi \otimes \psi \to \chi) \leftrightarrow (\phi \to \psi \to \chi)$ (*currying* and *uncurrying*)

(A6) $\phi \wedge \psi \leftrightarrow \phi \otimes (\phi \to \psi)$ (*weak conjunction*)

(A7) $\phi \wedge \psi \to \psi \wedge \phi$ (*commutativity of weak conjunction*)

(A8) $\phi \to \phi \vee \psi$ and $\psi \to \phi \vee \psi$ (*disjunction introduction*)

(A9) $(\phi \to \psi) \wedge (\chi \to \psi) \to \phi \vee \chi \to \psi$ (*disjunction elimination*)

(A10) $\bot \to \phi$ (*efq*)

A natural deduction system for **ILL**

$$\frac{}{\Gamma, \phi \vdash \phi}\ \text{Ax}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \phi \to \psi}{\Gamma, \Delta \vdash \psi} \to \text{E}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to \text{I}$$

$$\frac{\Gamma \vdash \phi \otimes \psi \qquad \Delta, \phi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi} \otimes \text{I}$$

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i} \wedge \text{E} \quad (i \in \{1, 2\})$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge \text{I}$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \vee \text{E}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} \vee \text{I} \quad (i \in \{1, 2\})$$

$$\frac{\Gamma, \phi, \phi \to \psi \vdash \chi}{\Gamma, \psi, \psi \to \phi \vdash \chi}\ \text{DIV}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} \bot \text{E}$$

Figure: Intuitionistic Łukasiewicz logic **ILL**

Theorem 4

The natural deduction system **ILL** *(Figure 6) is equivalent to the Hilbert-style system* $\mathbf{GBL}_{ewf}$ *of (2008), i.e.*

$$\Gamma \vdash_{\mathbf{ILL}} \phi \quad \textit{iff} \quad \bigotimes \Gamma \vdash_{\mathbf{GBL}_{ewf}} \phi$$

noting that we use $\psi \otimes \chi$ *where Bova and Montagna in (2009) use* $\psi \odot \chi$.

Proof.

Show the axioms of $\mathbf{GBL}_{ewf}$ are derivable in **ILL** and all the rules of **ILL** are derivable rules of $\mathbf{GBL}_{ewf}$. Straightforward. □

Algebraic semantics

- Before we introduce our Kripke semantics for **ILL**, we remind the audience about some standard algebraic semantics, for both intuitionistic Łukasiewicz logic and classical Łukasiewicz logic.
- In total we will consider four different classes of algebras: lattice-ordered monoids, residuated lattices, GBL-algebras, and MV-algebras.

Algebraic semantics cont'd

Definition 5 (Commutative Lattice-ordered monoid)

A structure $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 \rangle$ is a *commutative lattice-ordered monoid* if

- $\langle A, \wedge, \vee \rangle$ is a lattice
- $\langle A, \otimes, 1 \rangle$ is a commutative monoid
- $\otimes$ is monotonic increasing with respect to the lattice order on A.

Residuated Lattices

Definition 6 (Residuated lattice)

A structure $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1, \to \rangle$ is called a *residuated lattice* if

- $\langle A, \wedge, \vee, \otimes, 1 \rangle$ is a commutative lattice-ordered monoid
- $x \otimes y \leq z$ if and only if $x \leq y \to z$

Definition 7

A *GBL-algebra* is a residuated lattice which satisfies the *divisibility property*[a]: if $x \leq y$ then $y \otimes (y \rightarrow x) = x$. This is equivalent to require that the residuated lattice satisfies the equation:

$$x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x)$$

[a] Note that since $y \rightarrow x \leq y \rightarrow x$, it is always the case that $y \otimes (y \rightarrow x) \leq x$ (this is the counit of the adjunction defining residuation). The name “divisibility" property makes sense if one interprets $x \otimes y$ as multiplication $x \times y$, and $y \rightarrow x$ as division $\frac{x}{y}$. This is saying that if $0 \leq x \leq y \leq 1$ then $y \times \frac{x}{y} = x$. Note that if $y = 0$ then $x = 0$ as well.

Integral and Bounded GBL

Definition 8

A GBL-algebra is said to be *integral* if 1 is the top element of the lattice, i.e. $x \leq 1$, in which case we also denote 1 by $\top$.

Definition 9

A GBL-algebra is said to be *bounded* if the lattice has a bottom element $\bot$, i.e. $\bot \leq x$ for all $x \in A$.

Commutative, Bounded, Integral

Definition 10

Commutative, integral and bounded GBL-algebras provide an algebraic semantics for **ILL**. This is mentioned in various papers of Montagna et al, e.g. (2008) says "$\mathbf{GBL}_{ewf}$ is strongly algebraizable ... Its equivalent algebraic semantics is the variety of commutative, integral and bounded GBL-algebras".

Remark 1.1

We note that commutative, integral and bounded GBL-algebras are the same as Bounded pocrims with divisibility.

Bounded Pocrims

Definition 11

A bounded partially-ordered-commutative-residuated-integral-monoid, is

$$\mathcal{P} = \langle P, \leq, \top, \bot, \otimes, \rightarrow \rangle$$

P is a poset ordered by $\leq$, partial order given by

$$a \leq b \Leftrightarrow a \rightarrow b = \top$$

bounded by $\bot, \top$ with $\bot$ as the bottom and $\top$ as the top. Here $\otimes$ is a monoidal operation, with residual given by $\rightarrow$.

- $\langle P, \otimes, \top \rangle$ is a commutative monoid with neutral element $\top$;
- $\langle P, \leq \rangle$ is a partially ordered set such that $\leq$ is compatible with $\otimes$ (i.e., $a \leq b$ implies $a \otimes c \leq b \otimes c$ and $\top$ is the maximum of $\langle P, \leq \rangle$;
- $\langle P, \leq \rangle$ has the residuum property, that is $a \otimes c \leq b$ if and only if $c \leq a \rightarrow b$.

Valid Sequents in $\mathbf{GBL}_{ewf}$

Definition 12

A sequent $\phi_1, ..., \phi_n \vdash \psi$ is then said to be valid in $\mathcal{P}$, if $\phi_1 \otimes ... \otimes \phi_n \leq \psi$ holds in $\mathcal{P}$. A sequent is said to be *valid* if it is valid in *all* integral, commutative, GBL algebras. It is easy to show that the valid sequents, in the sense above, are precisely the ones provable in Intuitionistic Lukasiewicz logic.

Validity

Formally, given a commutative, integral and bounded GBL-algebras $\mathcal{A}$, and a mapping from propositional variables to elements of $\mathcal{A}$

$$p \quad \mapsto \quad [\![p]\!] \in A$$

we can extend that mapping to all formulas in the language of $\mathcal{L}$ in the straightforward way, e.g. $[\![\phi \otimes \psi]\!] := [\![\phi]\!] \otimes [\![\psi]\!]$. A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is then said to be valid in $\mathcal{A}$, if

$$[\![\phi_1]\!] \otimes \ldots \otimes [\![\phi_n]\!] \leq [\![\psi]\!]$$

holds in $\mathcal{A}$. A sequent is said to be valid if it is valid in all commutative, integral and bounded GBL-algebras. It is easy to show that the valid sequents, in the sense above, are precisely the ones provable in **ILL**.

MV-algebras

Definition 13 (MV-algebra)

An *MV-algebra* is a bounded, commutative, GBL-algebra where negation ($\neg x = x \to \bot$) is an involution, i.e. $(x \to \bot) \to \bot = x$, for all x.

MV-algebras

MV-algebras provide an algebraic semantics for (classical) Łukasiewicz logic. Here are are interested in a particular MV-algebra which we will use in our Kripke semantics for **ILL**.

MV chains

Definition

For $x \in [0, 1]$, let $\overline{x} := 1 - x$. The *standard MV-chain*, denoted $[0, 1]_{MV}$, is the *MV*-algebra defined as follows: The domain of $[0, 1]_{MV}$ is the unit interval $[0, 1]$, with the constants and binary operations defined as:

MV-chain

$$
\begin{aligned}
\top &:= 1 \\
\bot &:= 0 \\
x \wedge y &:= \min\{x, y\} \\
x \vee y &:= \max\{x, y\} \\
x \otimes y &:= \max\{0, \overline{\overline{x} + \overline{y}}\} \\
x \to y &:= \min\{1, \overline{\overline{y} - \overline{x}}\}
\end{aligned}
$$

note

The standard MV-chain is both an MV-algebra and a GBL-algebra. It is complete as a lattice.

note

$x \otimes y$ is equivalent to $\max\{0, x + y - 1\}$.To see this is so:

Proof.

$$
\begin{aligned}
x \otimes y &= \max\{0, \overline{\overline{x} + \overline{y}}\} \\
&= \max\{0, 1 - ((1 - x) + (1 - y))\} \\
&= \max\{0, 1 - (1 - x) - (1 - y)\} \\
&= \max\{0, 1 - 1 + x - 1 + y\} \\
&= \max\{0, x + y - 1\}
\end{aligned}
$$

□

Equivalents

note

$x \to y$ is equivalent to $\min\{1, y - x + 1\}$.

Proof.

$$
\begin{aligned}
x \to y &= \min\{1, \overline{\overline{y} - \overline{x}}\} \\
&= \min\{1, 1 - ((1 - x) - (1 - y))\} \\
&= \min\{1, 1 - (1 - x) + (1 - y)\} \\
&= \min\{1, 1 - 1 + x + 1 - y\} \\
&= \min\{1, y - x + 1\}
\end{aligned}
$$

□

Kripke Semantics for **ILL**

note

The Kripke semantics for **ILL** that we propose is based on the Bova-Montagna construction of poset sums introduced in (2008). We first need to define a particular class of functions from the set of worlds W to the standard MV-chain.

note

It may not appear to be a generalisation of Kripke semantics on first glance. The next few sections will make it clear that it is.

Kripke via Sloping functions

Definition 14

Let $\mathcal{W} = \langle W, \succeq \rangle$ be a partial order, and let $v \succ w := (v \succeq w) \wedge (v \neq w)$. A function $f : W \to [0, 1]_{MV}$ is said to be a *sloping function* if $f(w) > \bot$ implies $\forall v \succ w(f(v) = \top)$.

note

The above implies that if $f : W \to [0, 1]_{MV}$ is a sloping function and $f(w) < \top$ then $\forall v \prec w(f(v) = \bot)$. That is, along any increasing chain $w_1 \prec w_2 \prec \ldots \prec w_n$, there can only be at most one point i such that $\bot < f(w_i) < \top$, and for $j < i$ we must have $f(w_j) = \bot$, and for $j > i$ we must have $f(w_j) = \top$.

An Essential Lemma

Lemma 15

If $f\colon W \to [0,1]_{MV}$ *and* $g\colon W \to [0,1]_{MV}$ *are sloping functions, then the following functions are also sloping functions:*

$$
\begin{aligned}
(f \wedge g)(w) &:= \min\{fw, gw\} \\
(f \vee g)(w) &:= \max\{fw, gw\} \\
(f \otimes g)(w) &:= \max\{0, \overline{\overline{fw} + \overline{gw}}\}
\end{aligned}
$$

Proof of Lemma

Proof.

Let f, g be sloping functions. Let us consider each case:

- $f \wedge g$. Assume $(f \wedge g)(w) > \bot$, i.e. $\min\{fw, gw\} > \bot$. This implies that we have both $fw > \bot$ and $gw > \bot$. But since f and g are assumed to be sloping functions, we get that $\forall v \succ w(f(v) = \top)$ and $\forall v \succ w(g(v) = \top)$, from which it follows that $\forall v \succ w(\min\{f(v), g(v)\} = \top)$.
- $f \vee g$. Assume $(f \vee g)(w) > \bot$ i.e. $\max\{fw, gw\} > \bot$. This implies that we have at least one of $fw > \bot$ or $gw > \bot$. In case $fw > \bot$, f is a sloping function by hypothesis, so we have $\forall v \succ w(f(v) = \top)$ from which it follows $\forall v \succ w(\max\{f(v), g(v)\} = \top)$. The case of $gw > \bot$ is similar.
- $f \otimes g$. Assume $(f \otimes g)(w) > \bot$ i.e. $\max\{0, \overline{\overline{fw} + \overline{gw}}\} > 0$. This means $\max\{0, \overline{\overline{fw} + \overline{gw}}\} = \max\{0, f(w) + g(w) - 1\} > 0$; and hence $f(w) + g(w) - 1 > 0$.

Proof, continued

Proof.

($f \otimes g$ cont'd) This implies that neither $f(w) = \bot$ nor $g(w) = \bot$, i.e. we have both $f(w) > \bot$ and $g(w) > \bot$. Since both $f(w), g(w)$ are sloping functions by hypothesis $\forall v \succ w(f(v) = \top)$ and $\forall v \succ w(g(v) = \top)$. So $\forall v \succ w$ $\max\{0, f(v) + g(v) - 1\} = \max\{0, \top + \top - 1\} = \max\{0, \top + 0\} = \top$, as desired. □

BM- structures

Definition 16

A *Bova-Montagna structure* (or BM-structure) is a pair $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\text{BM}} \rangle$ where $\mathcal{W} = \langle W, \succeq \rangle$ is a poset, and $\Vdash^{\text{BM}}$ is an infix operator (on worlds and propositional variables) taking values in $[0, 1]_{MV}$, i.e. $(w \Vdash^{\text{BM}} p) \in [0, 1]_{MV}$, such that for any propositional variable p the function $\lambda w.(w \Vdash^{\text{BM}} p) \colon W \to [0, 1]_{MV}$ is a sloping function.

Floors and Infima

Definition 17

Let $\lfloor \cdot \rfloor$ be the usual "floor" operation, corresponding to the case distinction

$$\lfloor x \rfloor := \begin{cases} \top & \text{if } x = \top \\ \bot & \text{if } x < \top \end{cases}$$

Given a function $f\colon W \to [0,1]_{MV}$ and a $w \in W$, let us write $\mathbf{inf}_{v \succeq w}$ for the following construction:

$$\mathbf{inf}_{v \succeq w} f(v) \; := \; \min\{f(w), \inf_{v \succ w} \lfloor f(v) \rfloor\}$$

Infimum Sloping

Lemma 18

This definition of $\mathbf{inf}_{v \succeq w}$ *can also be equivalently written as*

$$\mathbf{inf}_{v \succeq w} f(v) := \begin{cases} f(w) & \textit{if } \forall v \succ w (f(v) = \top) \\ \bot & \textit{if } \exists v \succ w (f(v) < \top) \end{cases}$$

and for any $f : W \to [0, 1]$ *the function* $\lambda w.\mathbf{inf}_{v \succeq w} f(v)$ *is a sloping function.*

Proof.

First let us show that this is an equivalent definition. Consider two cases:

Case 1. $\forall v \succ w(f(v) = \top)$. In this case $\inf_{v \succ w}\lfloor f(v) \rfloor\} = \top$ and hence

$$\mathbf{inf}_{v \succeq w} f(v) \; = \; \min\{f(w), \top\} \; = \; f(w)$$

Case 2. $\exists v \succ w(f(v) < \top)$. In this case $\inf_{v \succ w}\lfloor f(v) \rfloor\} = \bot$

$$\mathbf{inf}_{v \succeq w} f(v) \; = \; \min\{f(w), \bot\} \; = \; \bot$$

In order to see that $\lambda w.\mathbf{inf}_{v \succeq w} f(v)$ is a sloping function, assume $\mathbf{inf}_{v \succeq w} f(v) > \bot$ for some w, and let $w' \succ w$. By definition we have that $\forall v \succ w(f(v) = \top)$, and hence $f(w') = \top$ and $\forall v \succ w'(f(v) = \top)$, which implies $\mathbf{inf}_{v \succeq w'} f(v) = \top$. □

Definition

Definition 19 (Kripke Semantics for $\mathcal{L}_\otimes$)

Given a BM-structure

$$\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$$

the valuation function $w \Vdash^{\mathrm{BM}} p$ on propositional variables p can be extended to all $\mathcal{L}_\otimes$-formulas as:

$$\begin{aligned}
w \Vdash^{\mathrm{BM}} \bot &:= \bot \\
w \Vdash^{\mathrm{BM}} \phi \wedge \psi &:= (w \Vdash^{\mathrm{BM}} \phi) \wedge (w \Vdash^{\mathrm{BM}} \psi) \\
w \Vdash^{\mathrm{BM}} \phi \vee \psi &:= (w \Vdash^{\mathrm{BM}} \phi) \vee (w \Vdash^{\mathrm{BM}} \psi) \\
w \Vdash^{\mathrm{BM}} \phi \otimes \psi &:= (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi) \\
w \Vdash^{\mathrm{BM}} \phi \to \psi &:= \mathbf{inf}_{v \succeq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi))
\end{aligned}$$

where the operations on the right-hand side are the operations on the standard MV-chain $[0,1]_{MV}$, and $\mathbf{inf}_{v \succeq w}$ as in Definition 17.

Yet another lemma

Lemma 20

For any formula ϕ the function $\lambda w.(w \Vdash^{\mathrm{BM}} \phi)\colon\ W \to [0,1]_{MV}$ is a sloping function.

Proof.

By induction on the complexity of the formula ϕ. The cases for $\psi \vee \xi, \psi \wedge \xi$ and $\psi \otimes \xi$ follow directly from Lemma 15. The case for $\psi \to \xi$ follows from Lemma 18. □

Generalised Monotonicity

We can now generalise the monotonicity property of intuitionistic logic to intuitionistic Łukasiewicz logic **ILL**:

Corollary 21 (Monotonicity)

The following (generalised) monotonicity property holds for all $\mathcal{L}_{\otimes}$-formulas ϕ, i.e.

$$\textit{if } v \succeq w \textit{ then } (v \Vdash^{K} \phi) \geq (w \Vdash^{K} \phi)$$

Proof.

This follows from the observation that sloping functions are in particular monotone functions. □

Validity with respect to BM

Definition 22

Let $\Gamma = \psi_1, \ldots, \psi_n$. Consider the following definitions:

- We say that a sequent $\Gamma \vdash \phi$ holds in a BM-structure $\mathcal{M}$ (written $\Gamma \Vdash^{\mathrm{BM}}_{\mathcal{M}} \phi$) if for all $w \in W$ we have

$$(w \Vdash^{\mathrm{BM}} \psi_1 \otimes \ldots \otimes \psi_n) \leq (w \Vdash^{\mathrm{BM}} \phi)$$

- A sequent $\Gamma \vdash \phi$ is said to be valid under the Kripke semantics for $\mathcal{L}_\otimes$ (written $\Gamma \Vdash^{\mathrm{BM}} \phi$) if $\Gamma \Vdash^{\mathrm{BM}}_{\mathcal{M}} \phi$ for all BM-structures $\mathcal{M}$.

Bova-Montagna structures generalise Kripke structures

Definition 23

Bova-Montagna structures generalise Kripke structures, i.e. Kripke structures are a particular case of BM-structures, when the valuations $w \models p \in [0,1]_{MV}$ are always in the finite set $\{\top, \bot\}$. These can then be identified with the Booleans. Therefore, any Kripke structure can be seen as a BM-structure, by defining

$$w \Vdash^{\mathrm{BM}} \phi = \begin{cases} \top & \text{if } w \Vdash^{\mathrm{K}} \phi \\ \bot & \text{if } w \nVdash^{\mathrm{K}} \phi \end{cases}$$

Recall that $\mathcal{L} \subset \mathcal{L}_{\otimes}$, so any $\mathcal{L}$-formula is also an $\mathcal{L}_{\otimes}$-formula.

Theorem 24

For any Kripke structure $\mathcal{K} = \langle \mathcal{W}, \Vdash^{\mathrm{K}} \rangle$ *and* $\mathcal{L}$*-formula* ϕ*, we have that*

$$w \Vdash^{\mathrm{K}} \phi \quad \textit{iff} \quad (w \Vdash^{\mathrm{BM}} \phi) = \top$$

Proof.

By induction on the complexity of the formula ϕ.

Basis: If ϕ is an atomic formulas the result is immediate.

Induction step: Suppose the result holds for all sub-formulas of ϕ:

Case 1. $\phi = \psi \wedge \chi$. We have:

$$\begin{aligned}
w \Vdash^{\mathrm{K}} \psi \wedge \chi \quad &\equiv \quad (w \Vdash^{\mathrm{K}} \psi) \wedge (w \Vdash^{\mathrm{K}} \chi) \\
&\overset{\text{(IH)}}{\Leftrightarrow} \quad (w \Vdash^{\mathrm{BM}} \psi) = \top \wedge (w \Vdash^{\mathrm{BM}} \chi) = \top \\
&\Leftrightarrow \quad \min\{w \Vdash^{\mathrm{BM}} \psi, w \Vdash^{\mathrm{BM}} \chi\} = \top \\
&\equiv \quad (w \Vdash^{\mathrm{BM}} \psi \wedge \chi) = \top
\end{aligned}$$

Proof.

Case 2. $\phi = \psi \vee \chi$. We have:

$$
\begin{aligned}
w \Vdash^{\mathrm{K}} \psi \vee \chi &\equiv (w \Vdash^{\mathrm{K}} \psi) \vee (w \Vdash^{\mathrm{K}} \chi) \\
&\overset{\text{(IH)}}{\Leftrightarrow} (w \Vdash^{\mathrm{BM}} \psi) = \top \vee (w \Vdash^{\mathrm{BM}} \chi) = \top \\
&\Leftrightarrow \max\{w \Vdash^{\mathrm{BM}} \psi, w \Vdash^{\mathrm{BM}} \chi\} = \top \\
&\equiv (w \Vdash^{\mathrm{BM}} \psi \vee \chi) = \top
\end{aligned}
$$

□

Proof.

Case 3. $\phi = \psi \to \chi$. Here we use that, when restricted to Kripke structures, $(v \Vdash^{\mathrm{BM}} \psi) \in \{\top, \bot\}$ and $(v \Vdash^{\mathrm{BM}} \chi) \in \{\top, \bot\}$, and hence

(i) $(v \Vdash^{\mathrm{BM}} \psi) = \top \to (v \Vdash^{\mathrm{BM}} \chi) = \top$ iff $(v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) = \top$

(ii) $\lfloor (v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) \rfloor = (v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi)$, i.e. the floor operation is unnecessary, and $\mathbf{inf}_{v \succeq w}$ becomes the standard $\inf_{v \succeq w}$ operation.

□

Proof.

Therefore:

$$
\begin{aligned}
w \Vdash^{\mathrm{K}} \psi \to \chi &\equiv \forall v \succeq w((v \Vdash^{\mathrm{K}} \psi) \to (v \Vdash^{\mathrm{K}} \chi)) \\
&\overset{(\mathrm{IH})}{\Leftrightarrow} \forall v \succeq w((v \Vdash^{\mathrm{BM}} \psi) = \top \to (v \Vdash^{\mathrm{BM}} \chi) = \top) \\
&\overset{(i)}{\Leftrightarrow} \forall v \succeq w((v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) = \top) \\
&\overset{(ii)}{\Leftrightarrow} \mathbf{inf}_{v \succeq w}((v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi)) = \top \\
&\equiv (w \Vdash^{\mathrm{BM}} \psi \to \chi) = \top
\end{aligned}
$$

which concludes the proof. □

Poset Products and BM structures

Definition 25

A *Poset Product* [2] (Def. 2) and [4] is defined over a poset $\mathcal{W} = \langle W, \succeq \rangle$. It is the algebra $\mathbf{A}_{\mathcal{W}}$ of signature $\mathcal{L}_{\otimes}$ whose elements are sloping functions $h \colon W \to [0,1]_{MV}$ and operations are defined as

$$
\begin{aligned}
(\bot)(w) &:= \bot \\
(h_1 \wedge h_2)(w) &:= \min\{h_1 w, h_2 w\} \\
(h_1 \vee h_2)(w) &:= \max\{h_1 w, h_2 w\} \\
(h_1 \otimes h_2)(w) &:= \max\{0, \overline{\overline{h_1 w} + \overline{h_2 w}}\} \\
(h_1 \to h_2)(w) &:= \begin{cases} h_1(w) \to h_2(w) & \text{if } \forall v \succ w(h_1(v) \leq h_2(v)) \\ \bot & \text{if } \exists v \succ w(h_1(v) > h_2(v)) \end{cases}
\end{aligned}
$$

Proof.

Since h_1 and h_2 are sloping functions, we have that

$$\forall v \succ w(h_1(v) \leq h_2(v)) \quad \Leftrightarrow \quad \forall v \succ w((h_1(v) \rightarrow h_2(v)) = \top)$$

Therefore, this last clause of the definition can be simplified to

$$(h_1 \rightarrow h_2)(w) \quad := \quad \mathbf{inf}_{v \succeq w}(h_1(v) \rightarrow h_2(v))$$

□

Poset Products and BM structures

Theorem 26

Poset Products and BM structures model the same theory.

Proof.

Let a poset $\mathcal{W} = \langle W, \succeq \rangle$ be fixed, and for each atomic formula p fix a sloping function $h_p \colon W \to [0,1]$. Any formula ϕ of $\mathcal{L}$ can be given an interpretation h_ϕ in the poset sum $\mathbf{A}_\mathcal{W}$ in the straightforward way. But with $\mathcal{W}$ and the family of functions h_p we can also define a BM-structure, taking

$$w \Vdash^{\mathrm{BM}}_{h} p \ :=_{[0,1]_{MV}} \ h_p(w)$$

It follows that these two interpretations of ϕ coincide, in the sense that

$$w \Vdash^{\mathrm{BM}} \phi \ = \ h_\phi(w)$$

The above can be shown by a straightforward induction on the complexity of ϕ. □

Corollary 27

One can always translate an interpretation of $\mathcal{L}_\otimes$ formulas in a poset product $\mathbf{A}_\mathcal{W}$ as a Kripke semantics (on the Kripke frame $\mathcal{W}$) for $\mathcal{L}_\otimes$ formulas, and vice-versa.

- We have extended the preceding results to the case of **BL**, or $\mathbf{GBL}_{ewf}$ with the axiom of prelinearity.
- The idea is to map formulas to a chain of MV-chains, via sloping functions. We call these LBM structures.
- We have similar results for Intuitionistic Affine logic, where we map formulas via monotone functions to involutive and bounded pocrims.

open problems

- Labelled calculi for $\mathbf{GBL}_{ewf}$, Intuitionistic Affine logic, even **BL**. This is current work.
- Applying these semantic insights to other logics in the neighborhood (e.g. **MTL**)
- Try and use the semantic insights to get better calculi for fuzzy logics?
- Direct proof of completeness.
- Multiple modal translations. . .[3]
- Monoidal T-norm logic, alias **MTL**, also has a poset product representation. We have identified a relational semantics for this, and this is present work.

Thank you!

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