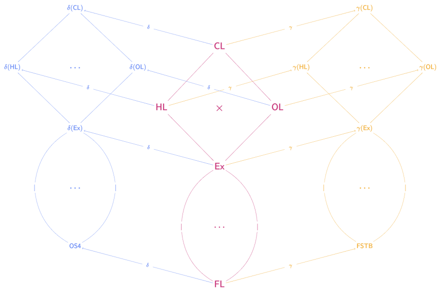
An Invitation to Fundamental Logic



Guillaume Massas

Scuola Normale Superiore di Pisa

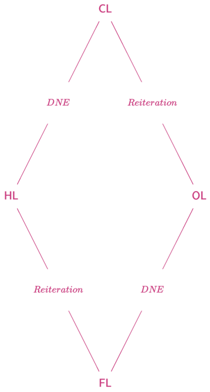
March 5, 2025

Outline

- My goal today is to give an overview of some recent results about fundamental logic, a weak propositional logic recently introduced by Holliday (2023).
- I will discuss two ongoing projects with Wes Holliday and Juan P. Aguilera.
- The first one is about modal translations of fundamental logic in the style of the Gödel-McKinsey-Tarski and Goldblatt translations for IPC and OL.
- The second is about the lattice of superfundamental logics and includes an axiomatization of orthointuitionistic logic.

Outline

Fundamental Logic



Fundamental Logic

- Holliday (2023) introduces a propositional logic that aims to capture exactly those properties of the connectives $\wedge$, $\vee$ and $\neg$ that follow from their introduction and elimination rules in Fitch's natural deduction system.
- The resulting logic, called fundamental logic, drops from Fitch's system for classical logic the rules of Reiteration and Double Negation Elimination.
- As such, fundamental logic generalizes both Heyting logic (the logic of the $\to$-free fragment of IPC) and orthologic.

Fundamental Logic and Natural Deduction

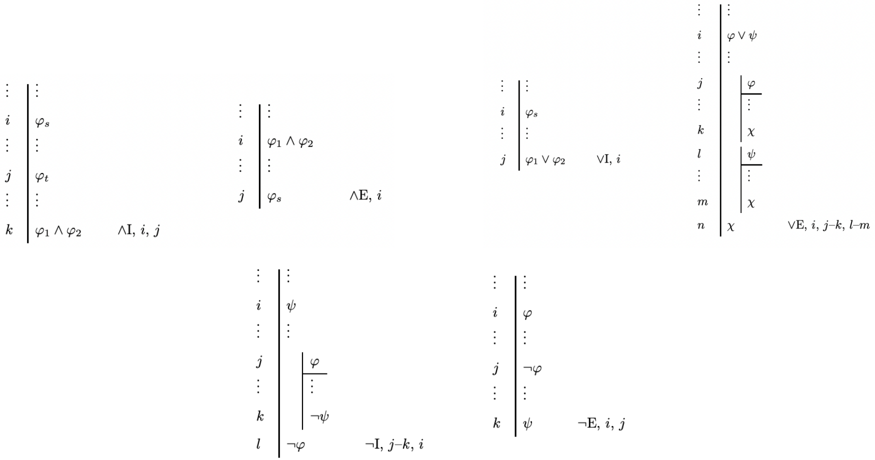
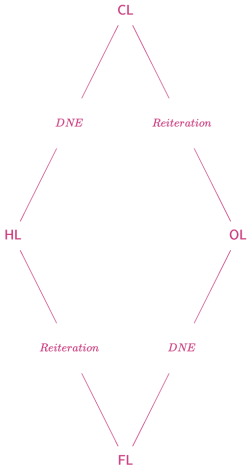


Figure 1: Rules of a Fitch-style system for FL

The Fundamental Diamond



A fundamental lattice is a pair $(L, \neg)$ such that L is a bounded lattice and $\neg : L \to L$ has the following two properties:

- $a \leq_L \neg b$ iff $b \leq_L \neg a$;
- $a \wedge \neg a = 0_L$.

Any ortholattice $(O, \neg)$ is a fundamental lattice satisfying the additional inequation

$$\neg\neg a \leq a.$$

If $(A, \neg, \to)$ is a Heyting algebra, its $\to$-free reduct, which I will call a Heyting lattice, is a distributive fundamental lattice that has the following residuation property:

$$a \wedge b \leq 0 \Leftrightarrow a \leq \neg b.$$

Theorem (Holliday 2023)

Fundamental lattices provide a sound and complete algebraic semantics for fundamental logic.

Let $\lhd$ be a relation on a set X, and consider the following two maps $\neg_{\lhd}, \neg_{\rhd} : \mathscr{P}(X) \to \mathscr{P}(X)$:

$$\neg_{\lhd} A = \{x \in X \mid \forall y \lhd x : y \notin A\}$$
$$\neg_{\rhd} A = \{x \in X \mid \forall y \rhd x : y \notin A\}.$$

The fixpoints of the maps $\neg_{\lhd}\neg_{\rhd}$ and $\neg_{\rhd}\neg_{\lhd}$ form two anti-isomorphic complete lattices $\chi_{\lhd}(X)$ and $\chi_{\rhd}(X)$, respectively called the positive and negative algebras of regular subsets of X.

The relation $\lhd$ also naturally induces two preorders on X:

- $x \leq_{\lhd} x'$ iff $\forall z \in X$: $z \lhd x \Rightarrow z \lhd x'$ (positive refinement);
- $x \leq_{\rhd} x'$ iff $\forall z \in X$: $x \rhd z \Rightarrow x' \rhd z$ (negative refinement).

Every positive regular set U is downward closed in the order $\leq_{\lhd}$. Every negative regular set V is downward closed in the order $\leq_{\rhd}$.

Definition

A fundamental frame is a pair $(X, \lhd)$ such that $\lhd \subseteq X \times X$ satisfies the following two conditions:

- Reflexivity: $x \lhd x$;
- Pseudo-symmetry: $y \lhd x$ implies that there is x' such that $x \geq_{\lhd} x' \lhd y$.

Theorem (Holliday 2023)

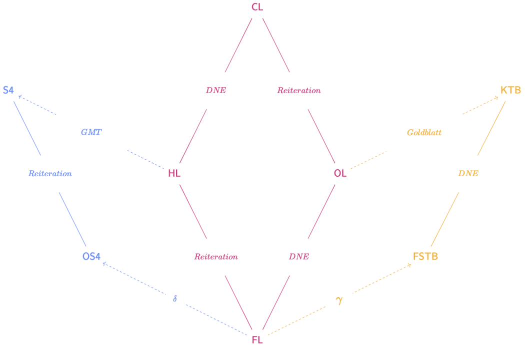
For any fundamental frame $(X, \lhd)$, the pair $(\chi_{\lhd}(X), \neg_{\lhd})$ is a fundamental lattice. Moreover, every fundamental lattice embeds into the positive algebra of some fundamental frame.

In particular, every fundamental lattice L embeds into the positive algebra of its reflexive canonical frame $(X_L, \lhd)$, where:

- Points in X_L are pairs $x = (x_F, x_I)$, with x_F, x_I a filter and an ideal on L respectively such that $x_F \cap x_I = \emptyset$ and $a \in x_F$ implies $\neg a \in x_I$.
- $(y_F, y_I) \lhd (x_F, x_I)$ iff $y_I \cap x_F = \emptyset$.

Two Translations

Join work with Wes Holliday



The GMT translation

- The celebrated Gödel-McKinsey-Tarski translation embeds IPC into S4 via the following map:
  - $\tau(p) := \Box p$;
  - $\tau(\phi \wedge \psi) := \tau(\phi) \wedge \tau(\psi)$;
  - $\tau(\phi \vee \psi) := \tau(\phi) \vee \tau(\psi)$;
  - $\tau(\phi \to \psi) := \Box(\tau(\phi) \to \tau(\psi))$.

Theorem (Gödel-McKinsey-Tarski)

The map τ is a full and faithful translation of IPC *into* S4*:*

$$\phi \vdash_{\mathsf{IPC}} \psi \Leftrightarrow \tau(\phi) \vdash_{\mathsf{S4}} \tau(\psi)$$

- Algebraically, the result relies on the fact that every Heyting algebra embeds into the $\Box$-fixpoints of an S4-modal algebra.

- Our goal is to extend this result beyond the distributive case, by working with orthologic as our background logic.

Definition

An OS4 lattice is a pair $(O, \Box)$ such that O is an ortholattice and $\Box : O \to O$ satisfies the following properties:

- $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$;
- $\Box a \leq a$;
- $\Box a \leq \Box\Box a$.

Clearly, $\Box$ is a closure operator on O for any OS4 lattice $(O, \Box)$. Moreover, the operation $\Box a \mapsto \Box\neg\Box a$ is a fundamental negation operation on the lattice of $\Box$ fixpoints.

Definition

An OS4-frame is a tuple $(X, \top, \leq)$ such that:

1. $\top$ is a reflexive and symmetric relation on X;
2. $\leq$ is a reflexive and transitive relation on X;
3. $\leq$ and $\top$ satisfy the following interaction condition:

$$z \top y \leq x \Rightarrow \exists x' \top x \, \forall x'' \top x' \, \exists y' : z \top y' \leq x''.$$

- The first condition guarantees that the positive algebra $\mathcal{C}_\top(X)$ induced by $\top$ is an ortholattice.
- The last two conditions ensure that $\Box_\leq U = \{x \in X \mid \forall y \leq x : y \in U\}$ defines an S4-operator on $\mathcal{C}_\top(X)$.

Definition

Given a OS4 frame $(X, \top, \leq)$, we define the openness relation $\lhd$ on X by

$$y \lhd x \Leftrightarrow \exists z : y \top z \leq x.$$

The fundamental reduct of $(X, \top, \leq)$ is the fundamental frame $(X, \lhd)$.

Lemma

Let $(X, \top, \leq)$ be an OS4-frame, and $(X, \lhd)$ its fundamental reduct. Then the following holds:

1. $\chi_{\lhd}(X) = ran(\Box_{\leq})$;
2. *The map $\Box_{\leq} : \mathcal{C}_{\top}(X) \to \chi_{\lhd}(X)$ is right adjoint to the inclusion $\iota : \chi_{\lhd}(X) \to \mathcal{C}_{\top}(X)$.*

Definition

Let $(X, \lhd)$ be a fundamental frame, and let $\mathrel{\lhd\!\!\rhd} = \lhd \cap \rhd$ be the symmetric kernel of $\lhd$.

1. We say that $(X, \lhd)$ is positively factoring if $y \lhd x$ implies $\exists z : y \mathrel{\lhd\!\!\rhd} z \leq_{\lhd} x$;
2. We say that $(X, \lhd)$ is negatively factoring if $y \lhd x$ implies $\exists z : x \mathrel{\lhd\!\!\rhd} z \leq_{\rhd} y$.

Finally, $(X, \lhd)$ is balanced if it is both positively and negatively factoring.

Lemma

Let $(X, \lhd)$ be a balanced fundamental frame. Then $(X, \mathrel{\lhd\!\!\rhd}, \leq_{\lhd})$ is an OS4 *frame with associated* OS4 *lattice $(\mathcal{C}_{\lhd\!\!\rhd}(X), \Box_{\leq_{\lhd}})$. Moreover, the following hold:*

1. $\chi_{\lhd}(X) \subseteq \mathcal{C}_{\lhd\!\!\rhd}(X)$;
2. *For any $A \in \mathcal{C}_{\lhd\!\!\rhd}(X)$, $\Box_{\leq_{\lhd}} \neg_{\lhd\!\!\rhd} A = \neg_{\lhd} A$;*
3. *The map $\Box_{\leq_{\lhd}}$ is right adjoint to the inclusion $\iota : \chi_{\lhd}(X) \to \mathcal{C}_{\lhd\!\!\rhd}(X)$, and $ran(\Box_{\leq_{\lhd}}) = \chi_{\lhd}(X)$.*

Embedding Fundmental Lattices into **OS4** Lattices

Theorem (Holliday and M. 2025)

For any fundamental lattice $(L, \neg_L)$, *there is a* OS4 *lattice* $(M, \neg_M, \Box_M)$ *and a lattice embedding* $e : L \to M$ *such that:*

1. *for any* $a \in L$, $\Box_M e(a) = e(a)$;
2. $e(\neg_L a) = \Box_M \neg_M e(a)$.

Proof.

The reflexive canonical frame $(X_L, \lhd)$ of any fundamental lattice is balanced. Letting $M = \mathcal{C}_{\oplus}(X_L)$, the composition of the Stone map $\widehat{\cdot} : L \to \chi_{\lhd}(X_L)$ with the inclusion $\iota : \chi_{\lhd}(X_L) \to \mathcal{C}_{\oplus}(X_L)$ is the required embedding. □

Fundamental Logic as "Constructive Logic for the Quantum Logician"

Let δ be the following translation of fundamental logic into OS4:

- $\delta(p) := \Box p$;
- $\delta(\phi \wedge \psi) := \delta(\phi) \wedge \delta(\psi)$;
- $\delta(\phi \vee \psi) := \delta(\phi) \vee \delta(\psi)$;
- $\delta(\neg\phi) := \Box\neg\delta(\phi)$.

Theorem (Holliday and M. 2025)

The translation δ is full and faithful:

$$\phi \vdash_{\mathsf{FL}} \psi \Leftrightarrow \delta(\phi) \vdash_{\mathsf{OS4}} \delta(\psi).$$

The Goldblatt Translation

- Goldblatt (1976) defined the following translation of OL into KTB (the logic of reflexive and symmetric Kripke frames):
  - $\sigma(p) := \Box\Diamond p$;
  - $\sigma(\phi \wedge \psi) := \sigma(\phi) \wedge \sigma(\psi)$;
  - $\sigma(\phi \vee \psi) := \Box\Diamond(\sigma(\phi) \vee \sigma(\psi))$;
  - $\sigma(\neg\phi) := \Box\neg\sigma(\phi)$.

Theorem (Goldblatt)

The map σ is a full and faithful translation of OL *into* KTB*:*

$$\phi \vdash_{\mathsf{OL}} \psi \Leftrightarrow \sigma(\phi) \vdash_{\mathsf{KTB}} \sigma(\psi)$$

- Algebraically, the result relies on the fact that every ortholattice embeds into the $\Box\Diamond$-fixpoints of a KTB-modal algebra.

- This time, we extend this result beyond ortholattices by working with IPC as our background logic.

Definition

An FSTB lattice is a triple $(A, \Box, \Diamond)$ such that A is a Heyting algebra, and $\Box$ and $\Diamond$ are unary operations on A satisfying the following axioms:

1. $\Box 1 = 1$, $\Diamond 0 = 0$;
2. $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$;
3. $\Diamond a \to \Box b \leq \Box(a \to b)$;
4. $\Diamond(a \to b) \leq \Box a \to \Diamond b$;
5. $\Box a \leq a \leq \Diamond a$;
6. $\Diamond\Box a \leq a \leq \Box\Diamond a$.

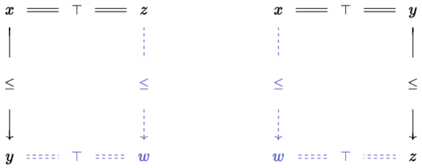
FSTB lattices are the analogue of KTB-algebras with Fischer Servi's modal logic as the base intuitionistic modal logic.

One can show that $\Box\Diamond$ is a closure operator on A for any FSTB lattice $(A, \Box, \Diamond)$. Moreover, the operation $\Box\Diamond a \mapsto \Box\neg\Box\Diamond a$ is a fundamental negation operation on the lattice of $\Box\Diamond$-fixpoints. Crucially, however, $\Box\Diamond a = \Box\neg\Box\neg a$ does not hold in general.

Definition

An FSTB-frame is a triple $(X, \leq, \top)$ such that:

1. $\leq$ is a quasi-order on X;
2. $\top$ is a reflexive and symmetric relation on X;
3. for any $x, y, z \in X$, $y \leq x \top z$ implies that there is $w \in X$ such that $y \top w \leq z$;
4. for any $x, y, z \in X$, $x \top y \geq z$ implies that there is $w \in X$ such that $x \geq w \top z$.



Given an FSTB-frame $(X, \leq, \top)$, we let the complex algebra of $(X, \leq, \top)$ be the FSTB lattice $Dn(X)$ of downsets of $(X, \leq)$ endowed with the operations $\Box_\top$ and $\Diamond_\top$ defined by

$$\Box_\top A = \{x \in X \mid \forall y, z : x \geq y \top z \Rightarrow z \in A\}$$

and

$$\Diamond_\top A = \{x \in X \mid \exists y \in A : x \top y\}.$$

Moreover, we define the openness relation $\lhd$ on $(X, \leq, \top)$ as follows:

$$y \lhd x \Leftrightarrow \exists z : y \top z \leq x.$$

Lemma

For any FSTB-frame $(X, \leq, \top)$, its fundamental reduct $(X, \lhd)$ is a fundamental frame. Moreover, the positive algebra of $(X, \lhd)$ coincides with the fixpoints of the $\Box_\top \Diamond_\top$ operator on $Dn(X)$, and $\neg_\lhd A = \Box_\top \neg A$ for any $A \in Dn(X)$.

Strongly Factoring Fundamental Frames

Definition

Let $(X, \lhd)$ be a fundamental frame. We say that y strongly refines x (denoted $y \preccurlyeq_{\lhd} x$) if $y \leq_{\lhd} x$ and $x \leq_{\rhd} y$.

A fundamental frame $(X, \lhd)$ is strongly factoring if $y \lhd x$ implies $\exists z : y \mathrel{\lhd\!\!\rhd} z \preccurlyeq_{\lhd} x$.

Lemma

For any fundamental frame $(X, \lhd)$, its modal companion $(X, \preccurlyeq_{\lhd}, \mathrel{\lhd\!\!\rhd})$ is an FSTB-frame. Moreover, $(\chi_{\lhd}(X), \neg_{\lhd})$ is isomorphic to the $\Box_{\lhd\!\!\rhd}\Diamond_{\lhd\!\!\rhd}$-fixpoints of $Dn_{\preccurlyeq_{\lhd}}(X)$, endowed with the operation $\Box_{\lhd\!\!\rhd}\neg_{\preccurlyeq_{\lhd}}$.

Embedding Fundamental Lattices into FSTB Lattices

Theorem (Holliday and M. 2025)

For any fundamental lattice $(L, \neg_L)$*, there is an* FSTB *lattice* $(A, \Box, \Diamond)$ *and a meet-semilattice embedding* $\nu : L \to A$ *such that, for any* $a, b \in L$*:*

- $\Box\Diamond\nu(a) = a$;
- $\nu(a \vee b) = \Box\Diamond(\nu(a) \vee \nu(b))$;
- $\nu(\neg_L a) = \Box\neg\nu(a)$.

Proof.

The reflexive canonical frame $(X_L, \lhd)$ of L is strongly factoring. Consequently, letting $(A, \Box, \Diamond)$ be $(Dn_{\preccurlyeq_\lhd}(X), \Box_{\Leftrightarrow}, \Diamond_{\Leftrightarrow})$, the required embedding is given by the composition of the Stone map $\widehat{\cdot} : L \to \chi_\lhd(X_L)$ with the inclusion map $\iota : \chi_\lhd(X_L) \to Dn_{\preccurlyeq_\lhd}(X)$. $\Box$

Fundamental Logic as "Quantum Logic for the Constructive Logician"

Let γ be the following translation of fundamental logic into FSTB:

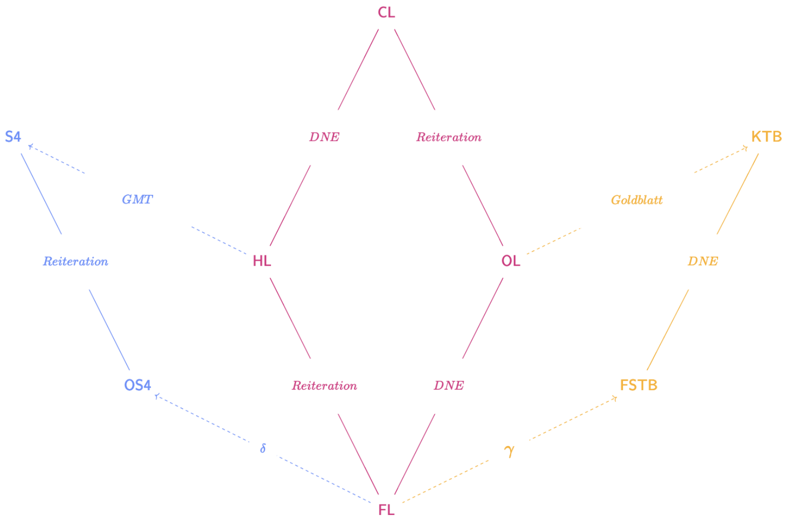
- $\gamma(p) := \Box\Diamond p$;
- $\gamma(\phi \wedge \psi) := \gamma(\phi) \wedge \gamma(\psi)$;
- $\gamma(\phi \vee \psi) := \Box\Diamond(\gamma(\phi) \vee \gamma(\psi))$;
- $\gamma(\neg\phi) := \Box\neg\gamma(\phi)$.

Theorem (Holliday and M. 2025)

The translation γ is full and faithful:

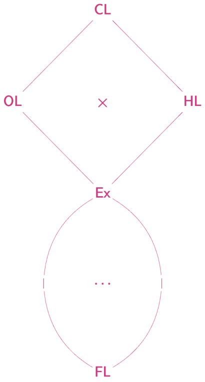
$$\phi \vdash_{\mathsf{FL}} \psi \Leftrightarrow \gamma(\phi) \vdash_{\mathsf{FSTB}} \gamma(\psi).$$

The Fundamental Lotus



Superfundamental Logics

joint work with Juan P. Aguilera



Fundamental Logic and Orthointuitionistic Logic

The following are examples of inequations that are valid in any ortholattice and in any Heyting lattice:

$$\neg\neg a \wedge \neg\neg b \leq \neg\neg(a \wedge b); \tag{Nu}$$

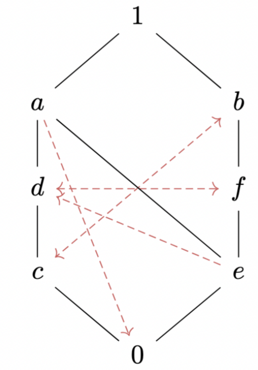
$$\neg\neg a \wedge b \wedge (c \vee d) \leq a \vee (b \wedge c) \vee (b \wedge d); \tag{Vi}$$

$$\neg(a \wedge ((b \wedge c) \vee (b \wedge d))) \wedge a \leq (b \wedge (c \vee d)) \vee \neg(b \wedge (c \vee d)). \tag{Cl}$$

None of these, however, is valid in fundamental logic.

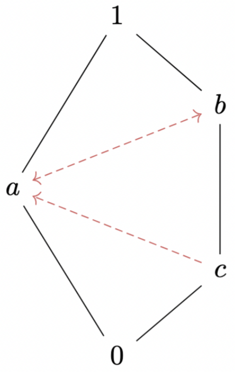
A Counterexample to (Nu)

$$\neg\neg a \wedge \neg\neg b \leq \neg\neg(a \wedge b) \tag{Nu}$$



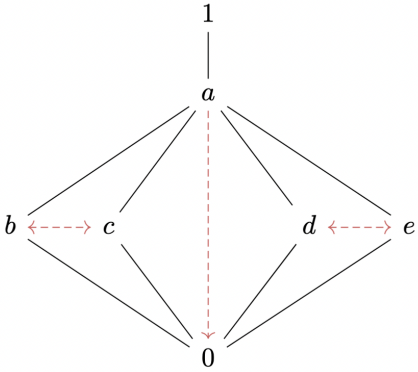
A Counterexample to (Vi)

$$\neg\neg c \wedge b \wedge (a \vee c) \leq c \vee (b \wedge a) \vee (b \wedge c) \qquad \text{(Vi)}$$



A Counterexample to (CI)

$$\neg(1 \wedge ((b \wedge c) \vee (b \wedge d)) \wedge 1 \leq (b \wedge (c \vee d)) \vee \neg(b \wedge (c \vee d)) \qquad \text{(CI)}$$



Definition

An Ex-lattice is a fundamental lattice in which the following axiom is valid:

$$\neg\Big[a \wedge \Big((b \wedge c) \vee (b \wedge d)\Big)\Big] \wedge a \wedge (c \vee e) \wedge \neg\neg f \leq \neg\neg(a \wedge f) \wedge \Big[(a \wedge c) \vee (a \wedge e) \vee f\Big] \wedge \Big[\Big(b \wedge (c \vee d)\Big) \vee \neg\Big(b \wedge (c \vee d)\Big)\Big]. \tag{Ex}$$

- Ex-Lattices generalize both ortholattices and Heyting lattices.
- Moreover, any Ex-lattice satisfies (Nu), (Vi) and (Cl).

Theorem (Aguilera and M.)

Any Ex-lattice L embeds into the cartesian product $O_L \times A_L$ of an ortholattice O_L and a Heyting lattice A_L.

Proof.

In fact, we show that any lattice L satisfying (Nu), (Vi) and (Cl) embeds into a lattice of the form $O_L \times A_L$.

- Step 1: Define an equivalence relation $\sim$ on L by letting $a \sim b$ iff $\neg a = \neg b$. By (Nu), $\sim$ is a congruence relation on L, so we let O_L be the quotient ortholattice determined by $\sim$.
- Step 2: Let $Pf(L)$ be the set of all prime filters on L. By (Cl), the Stone map $a \mapsto \widehat{a} = \{P \in Pf(L) \mid a \in P\}$ is a fundamental lattice homomorphism. We let A_L be the lattice of downsets of $Pf(L)$ generated by the range of this map.
- Step 3: We define $e : L \to O_L \times A_L$ by $e(a) = (a^{\sim}, \widehat{a})$. By (Vi), whenever $\neg a = \neg b$ and $a \nleq b$, there is $P \in Pf(L)$ such that $a \in P$ and $b \notin P$. This shows that e is injective.

□

Corollary

The logic Ex = FL + (Ex) *coincides with the intersection of* OL *and* HL*.*

Proof.

Clearly, Ex $\subseteq$ OL $\cap$ HL. For the converse direction, suppose that $\phi \vdash \psi$ is not derivable in Ex. Then there is an Ex-lattice L such that $L \not\models \phi \leq \psi$. By the Ex-embedding theorem, there is an ortholattice O_L, a Heyting lattice A_L and an embedding $e : L \to O_L \times A_L$. But this means that $O_L \not\models \phi \leq \psi$ or $A_L \not\models \phi \leq \psi$. Either way, $\phi \vdash \psi$ is not derivable in OL $\cap$ HL. □

Note that Ex is also axiomatized by

$$\mathsf{FL} + (\mathsf{Nu}) + (\mathsf{Vi}) + (\mathsf{Cl}).$$

Moreover, the result generalizes in the following way.

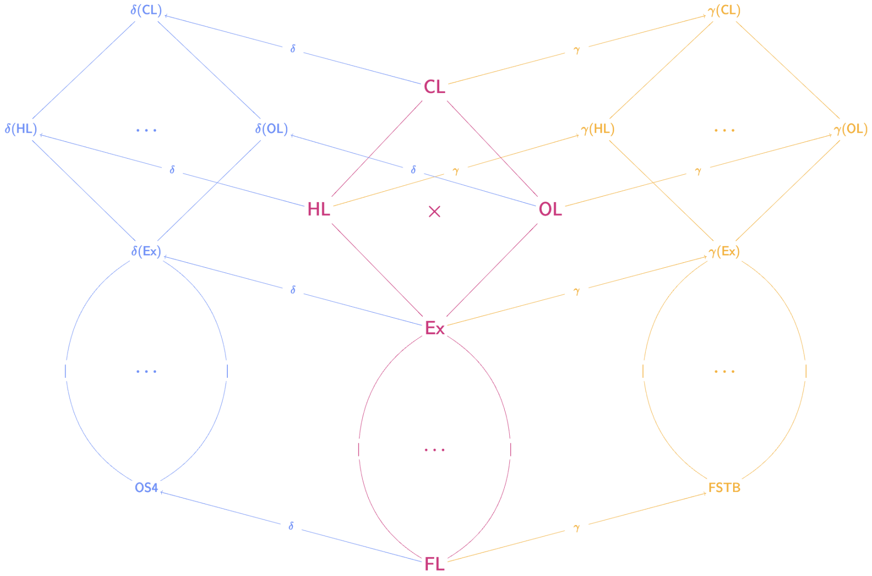
Theorem (Aguilera and M.)

Let $L_{\mathcal{O}}$ and $L_{\mathcal{I}}$ be extensions of OL *and* HL, *axiomatized by $\{\phi_i \vdash \psi_i\}_{i \in I}$ and $\{\phi_j \vdash \psi_j\}_{j \in J}$ respectively. Then $L_{\mathcal{O}} \cap L_{\mathcal{I}}$ is axiomatized by*

$$\mathsf{Ex} + \{\neg\neg\phi_i \vdash \neg\neg\psi_i\}_{i \in I} + \{\phi_j \vdash \psi_j \vee \neg\psi_j\}_{j \in J}.$$

*Moreover, the map $(L_{\mathcal{O}}, L_{\mathcal{I}}) \mapsto L_{\mathcal{O}} \cap L_{\mathcal{I}}$ is an order-isomorphism between the lattice of super-*Ex *logics and the lattice* **SO** $\times$ **SH**, *i.e., the cartesian product of the lattice of quantum logics with the lattice of super-Heyting logics.*

Summing Up



Open Problems

- What more can be said about δ and γ modal companions of superfundamental logics?
- Is there a version of the Blok-Esakia theorem that holds for the δ translation? For example, is there an isomorphism between (a subset of) the interval $[\mathsf{OS4}, \mathsf{OS5}]$ and (a subset of) the interval $[\mathsf{FL}, \mathsf{OL}]$?
- What is the complexity of the consequence relation in Ex-logic? Existing results tell us that this is at most co-NP complete. Is this bound optimal?
- Is there an elegant, cut-free sequent calculus for Ex-logic?