On provability logic of Heyting Arithmetic

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• □ as provability.
• Gödel 1933: Based on BHK.
PL(T) := Provability logic of $T := \{ A \in \mathcal{L}_{\Box} : \forall \sigma \ T \vdash \sigma_T A \}$
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- \sigma_T(p) := \sigma(p) for atomics.
- \sigma_T commutes with boolean connectives.
- \sigma_T(\Box A) := Pr_T(\neg \sigma_T A\neg).
The Provability logic of PA is GL

- All theorems of classical propositional logic.
- $K := \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.
- L"ob := $\Box(\Box A \rightarrow A) \rightarrow \Box A$. Implies $\Box A \rightarrow \Box \Box A$.
- modus ponens: $A, A \rightarrow B/B$.
- Necessitation: $A/\Box A$. 
GL is sound and complete for
finite transitive irreflexive Kripke models.
Solovay’s proof (roughly)

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- $GL \not
not \vDash A$.
- $K, w \not\vDash A$.
- $f: \mathbb{N} \rightarrow W$ and $f(0) := \text{root}$.
- $f(n) \prec f(n + 1)$ iff $n + 1$ proves this fact that $f$ will not remain at $f(n + 1)$.
- $\sigma(p) := \bigvee_{w \vDash p}(\text{lim } f = w)$.
- $PA \not
not \vDash \sigma_T(A)$. 
\[ \text{PL}_\Sigma(T) := \Sigma_1\text{-Provability logic of } T := \{ A \in L_{\square} : \forall \sigma \ T \vdash \sigma_T A \} \]

**Theorem (Visser)**

\[ \text{PL}_\Sigma(\text{PA}) = \text{GLC}_a := \text{GL} + p \rightarrow \Box p \text{ for atomic } p \text{'s}. \]

**Proof.**

Similar to the original proof, except for

- \[ \sigma(p) := \bigvee_{w \models p} (\exists x f(x) = w). \]
Theorem (Ardeshir & M. 2015)

One may reduce the arithmetical completeness of GL to the one for GLCa.

Proof.

Let GL $\nvdash A$. Then find a Kripke counter model of $A$. Then transform it to a Kripke model of GLCa which refutes $\alpha(A)$ for some propositional substitution $\alpha$. Thus GLCa $\nvdash \alpha(A)$. Finally use arithmetical completeness of GLCa and obtain $\sigma$ such that PA $\nvdash \sigma\alpha(A)$. 

$\square$
• Relative provability logics: \( \text{PL}(T, S) \).
Generalizations

- Relative provability logics: $PL(T, S)$.

- Poly-modal Provability Logic.
  Gaparidze, Beklemishev, Pakhomov, Bezhanishvili, Icard, Gabelaia and ... (1986-)

- Interpretability logic. $A \triangleright B$
  Visser, Berarducci, de Jongh, Veltman, Shavrukov and ... (1980-1990)

- Provability logic of weak systems of arithmetic (bounded arithmetic).
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- Provability logic of weak systems of arithmetic (bounded arithmetic).
- Provability logic of Heyting Arithmetic HA.
A. Visser 1980 first considered this. Since then many partial related results where obtained. We review them later. Main source for difficulty: HA-verifiable admissible rules.

\[
\neg A \rightarrow (B \lor C) \\
\frac{(\neg A \rightarrow B) \lor (\neg A \rightarrow C)}{}
\]
Admissible rules

- \( A \sim T B \) iff \( \forall \alpha (T \vdash \alpha(A) \Rightarrow T \vdash \alpha(B)) \).
- Example: \( \neg A \rightarrow (B \lor C) \sim (\neg A \rightarrow B) \lor (\neg A \rightarrow C) \).
- In the provability logic of HA, the above rule reflected as:
  \[ \Box(\neg A \rightarrow (B \lor C)) \rightarrow \Box((\neg A \rightarrow B) \lor (\neg A \rightarrow C)). \]

- Why not classically interesting?

  \( A \sim_{\text{CPC}} B \) iff \( \text{CPC} \vdash A \rightarrow B \).
Admissible rules of IPC

- For every $A \sim_{IPC} B$ we have $\Box A \rightarrow \Box B$ in PL(HA).
- What are the admissible rules of IPC? Decidable? (H. Friedman 1975)
For every $A \vdash_{\text{IPC}} B$ we have $\Box A \rightarrow \Box B$ in $\text{PL}(\text{HA})$.

What are the admissible rules of IPC? Decidable? (H. Friedman 1975)
The system $[T, \Delta]$

Axioms: Define $\{A\}_\Delta(E) := \begin{cases} E : E \in \Delta \\ A \rightarrow E : \text{otherwise} \end{cases}$

$$\frac{T \vdash A \rightarrow B}{A \triangleright B} \quad [T]$$

$$A = \bigwedge_{i=1}^{n}(E_i \rightarrow F_i) \quad B = \bigvee_{i=n+1}^{n+m}(F_i)$$

$$\frac{(A \rightarrow B) \triangleright \bigvee_{i=1}^{n+m} \{A\}_\Delta(E_i)}{V(\Delta)}$$

Rules:

$$\frac{A \triangleright B \quad A \triangleright C}{A \triangleright (B \land C)} \quad \text{Conj} \quad \frac{A \triangleright B \quad B \triangleright C}{A \triangleright C \quad B \triangleright C}{A \triangleright (B \lor C)} \quad \text{Disj} \quad \frac{A \triangleright B \quad (D \in \Delta)}{(D \rightarrow A) \triangleright (D \rightarrow B)} \quad \text{Mont}(\Delta)$$

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Theorem (Iemhoff 2001)

\[ A \not\sim_{IPC} B \iff [\text{IPC}, \{\top, \bot\}] \vdash A \triangleright B. \]

Theorem (Visser 2002)

\[ A \not\sim_{IPC} B \iff [\text{IPC}, \{\top, \bot\}] \vdash A \triangleright B \iff \square A \rightarrow \square B \in PL(\text{HA}). \]
DP means that if a disjunction is derivable, then either of them are derivable.

IPC, IQC and HA has DP.

$\text{CPC} \vdash p \lor \neg p$ while $\text{CPC} \not\vdash p$ and $\text{CPC} \not\vdash \neg p$.

$\Box(A \lor B) \rightarrow (\Box A \lor \Box B) \in \text{PL(HA)}$?

H. Friedman 1975: No!

D. Leivant 1975: $\Box(A \lor B) \rightarrow \Box(\Box A \lor \Box B) \in \text{PL(HA)}$.

Above axiom together with reflection implies DP.
\[ \forall S \in \Sigma_1 \ (\text{HA} \vdash \neg \neg S \ \text{implies} \ \text{HA} \vdash S). \]
∀S ∈ Σ₁ (HA ⊢ ¬¬S implies HA ⊢ S).

Theorem (Visser 1981)

□¬¬□A → □□A ∈ PL(HA).

Theorem (Visser 1981)

The letterless fragment of PL(HA) is decidable.
Let us define the Leivant’s axiom schema as follows:

\[(\text{Le}): A \triangleright \Box A \text{ for every } A \text{ and } B.\]

**Theorem (M. 2022)**

\[\text{iGLH} := \text{iGL} + \{\Box A \to \Box B : [\text{iGL}, \Box] \text{Le} \vdash A \triangleright B\} = \text{PL(HA)}.\]
Let us define the Leivant’s axiom schema as follows:

\((\text{Le})\): \( A \triangleright \Box A \) for every \( A \) and \( B \).

**Theorem (M. 2022)**

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iGLH := iGL + \{ \Box A \rightarrow \Box B : [iGL, \Box] \text{Le} \vdash A \triangleright B \} = PL(\text{HA}).
\]

**Theorem (Ardeshir & M. 2018)**

\[
iGLC_\sigma H_\sigma := iGLC_\sigma + \{ \Box A \rightarrow \Box B : [iGLC_\sigma, \text{atomb}] \text{Le} \vdash A \triangleright B \} = PL_{\Sigma}(\text{HA})
\]
The arithmetical soundness of this system in a more general setting, namely $\Sigma_1$-preservativity, was already known by Visser, de Jongh and Iemhoff (2001).
1. Let \( i\text{GLH} \not\models A \).
2. Find some \( \alpha \) s.t. \( i\text{GLC}_aH\sigma \not\models \alpha(A) \).
3. Use arithmetical completeness of \( i\text{GLC}_aH\sigma \) to find \( \sigma \) s.t. \( HA \not\models \sigma\alpha(A) \).
We first need a finite, or at least well-behaved Kripke semantics.

Iemhoff already provided a semantic for an extension of $iGLH$ in the language with binary modal operator.

Iemhoff’s semantics are not finite.

At least we failed to use it for the purpose of reduction.

We provided a finite *mixed* semantic which is a combination of derivability and Kripke-style validity.

It well fits for preservativity.
\( A \rhd^\Gamma B \text{ iff for every } E \in \Gamma \ (T \vdash E \rightarrow A \text{ implies } T \vdash E \rightarrow B) \)
Preservativity

\[ A \vDash_{\Gamma} B \text{ iff for every } E \in \Gamma (T \vdash E \to A \text{ implies } T \vdash E \to B) \]

**Theorem (M. 2022)**

\[ \llbracket \text{iGL, } \Box \rrbracket L \vdash A \triangleright B \text{ iff } A \vDash_{\text{iGL}} B. \]

\[ \Gamma := C\downarrow \text{SN(}\Box\text{)} \]

Roughly, \( \Gamma \) is the set of modal propositions which could be projected to a \( \text{NNIL} \)-proposition.
It is a relativised version of Ghilardi’s unification for IPC. (1999)
Projectivity: standard definition

A is projective iff there is some $\theta$ s.t. $\text{IPC} \vdash \theta(A)$ and $A \vdash_{\text{IPC}} \theta(x) \leftrightarrow x$ for every variable $x$.

**Theorem**

A projective unifier is a most general unifier.

**Proof.**

Consider some $\alpha$ s.t. $\text{IPC} \vdash \alpha(A)$. Then $\alpha(A) \vdash \alpha\theta(x) \leftrightarrow \alpha(x)$. This means that $\alpha\theta = \theta$, hence $\theta$ is more general than $\alpha$.

**Theorem (Ghilardi 1999)**

For every $A$ there is a best approximation of $A$ by finite disjunctions of projective propositions $\bigvee \Pi(A)$. Moreover

$$A \sim_{\text{IPC}} \bigvee \Pi(A)$$
A is NNIL(par)-projective if there is some $\theta$ and $B \in \text{NNIL(par)}$ s.t. $\text{IPC} \vdash \theta(A) \leftrightarrow B$ and $A \vdash_{\text{IPC}} \theta(x) \leftrightarrow x$ for every var $x$.

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For every $A$ there is a best approximation of $A$ by finite disjunctions of projective propositions $\bigvee \Pi(A)$. Moreover

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**Theorem (M. 2022)**

$A \equiv_{\text{IPC}}^\text{N(par)} B$ iff $[\text{IPC, NNIL(par)}] \vdash A \triangleright B$ iff $A \equiv_{\text{IGL}}^{\text{SN(par)}} B$. 
Axiomatizing modal preservativity

Theorem

\[
[iGL, \text{parb}] \vdash A \triangleright B \iff A \approxeq_{iGL}^{CSN(\Box)} B.
\]
\[ iGLH(\Gamma, T) := iGL + \{\Box A \rightarrow \Box B : A \vdash_{\Gamma T} B\} \]
Roughly speaking, a mixed semantic is a usual Kripke model for intuitionistic modal logic, which is augmented by a family of propositions \( \{ \varphi_w \}_w \in W \) with:

- \( \varphi_w \in \Gamma \),
- \( K, w \models \varphi_w \),
- \( K, w \models \Box A \) iff for every \( u \sqsupseteq w \) we have \( T, \Delta_w, \varphi_u \models A \).
Thanks For Your Attention