Coalgebras and Corecursive Algebras in Continuous Mathematics

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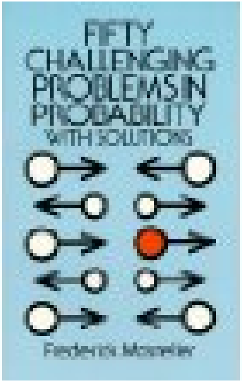
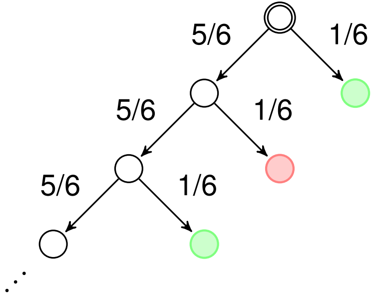
ILLC LLAMA Seminar
February 14, 2024

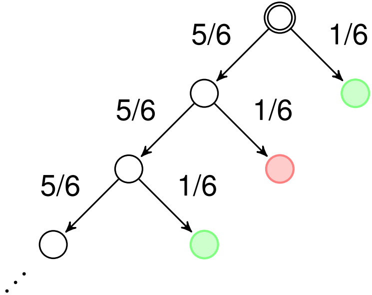
Frederick Mosteller,
Fifty Challenging Problems in Probability With Solutions



Problem 4: If one throws a die repeatedly, starting with roll 1, what is the probability that the first 6 is on an odd numbered roll?

We have a Markov chain with success and fail nodes at the end





So the total probability for a success is

$$\frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \cdots \quad = \quad \frac{6}{11}$$

Modifying Mosteller's text a bit vague, but it is still inspiring

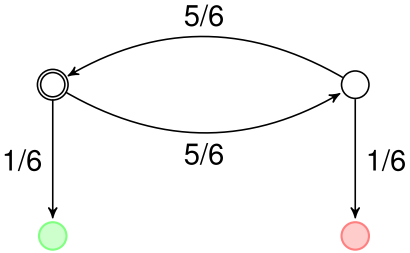
"But a beautiful way to solve the problem is as follows:
To get the first 6 on an odd numbered roll,
one can either get it on the first roll,
or else fail to get a 6 on the first roll, and
then get the first 6 on an *even* numbered roll after that."

Let p be the probability that when we roll repeatedly,
the first 6 is on an odd numbered roll.

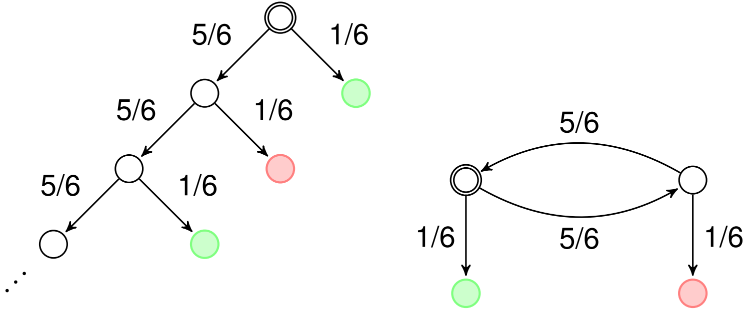
Let q be the probability that when we roll repeatedly,
the first 6 is on an even numbered roll.

$$p = \frac{1}{6} + \frac{5}{6}q$$

$$q = 1 - p$$


5/6
5/6
1/6
1/6

But what exactly is the relation between the two pictures?



For us, this is the important question.

Is the relationship describable in terms that we understand from other areas?

Let $\delta < 1$.
Let $r_0, r_1, \ldots, r_n, \ldots$ be any sequence of elements of $[0, 1 - \delta]$.

Then there is a unique sum

$$\sum_{n=0}^{\infty} r_n \delta^n = r_0 + r_1\delta + \cdots + r_n\delta^n + \cdots$$

Proof 1: algebraic mathematics

Recall that the infinite sum above is really
the limit of the sequence of finite partial sums

$$r_0, \quad r_0 + r_1\delta, \quad \ldots, \quad r_0 + r_1\delta + \cdots + r_n\delta^n, \quad \ldots$$

This is a Cauchy sequence, etc.

Let $X = \{x_0, x_1, \ldots\}$ be a set of variables.

Let's solve the infinite system of equations

$$\begin{aligned} x_0 &= r_0 + \delta \cdot x_1 \\ x_1 &= r_1 + \delta \cdot x_2 \\ &\vdots \\ x_n &= r_n + \delta \cdot x_{n+1} \\ &\vdots \end{aligned}$$

Forget δ for a moment, and regard the system as a map

$$\langle r, \mathrm{next} \rangle : X \to [0, 1 - \delta] \times X.$$

(For example, $r(x_0) = r_0$, and $\mathrm{next}(x_0) = x_1$.)

Given $\langle r, \mathrm{next} \rangle$, we are after a map *solution* $: X \to [0, 1]$ so that

$$\mathit{solution}(x) = r(x) + \delta \cdot \mathit{solution}(\mathrm{next}(x))$$

Let $X = \{x_0, x_1, \ldots\}$ be a set of variables.

Let's solve the infinite system of equations

$$\begin{array}{rclcrcl} x_0 & = & r_0 + \delta \cdot x_1 & \quad & \mathit{solution}(x_0) & = & r_0 + \delta \cdot \mathit{solution}(x_1) \\ x_1 & = & r_1 + \delta \cdot x_2 & & \mathit{solution}(x_1) & = & r_1 + \delta \cdot \mathit{solution}(x_2) \\ & \vdots & & & & \vdots & \\ x_n & = & r_n + \delta \cdot x_{n+1} & & \mathit{solution}(x_n) & = & r_n + \delta \cdot \mathit{solution}(x_{n+1}) \\ & \vdots & & & & \vdots & \end{array}$$

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Given $\langle r, \mathrm{next} \rangle$, we are after a map $\mathit{solution} : X \to [0, 1]$ so that

$$\mathit{solution}(x) = r(x) + \delta \cdot \mathit{solution}(\mathrm{next}(x))$$

A beautiful way to think of

$$\sum_{n=0}^{\infty} r_n \delta^n$$

is that it is r_0 added to $\sum_{n=1}^{\infty} r_n \delta^n$.

Of course $\sum_{n=1}^{\infty} r_n \delta^n$ is r_1 added to $\sum_{n=2}^{\infty} r_n \delta^n$.

etc.

What is going on

Instead of thinking of evaluating one infinite sum
we are solving an infinite system of (simple) equations.

LEMMA

For all maps $\langle r, next\rangle$*, there is a unique map* *solution*:

$$\begin{array}{ccc} X & \xrightarrow{\langle r,next\rangle} & [0,1-\delta]\times X \\ {\scriptstyle solution}\downarrow & & \downarrow{\scriptstyle [0,1-\delta]\times solution} \\ [0,1] & \xleftarrow[a]{} & [0,1-\delta]\times[0,1] \end{array}$$

where $a(x,y) = x + \delta \cdot y$.

PROOF.

The function set $Hom(X,[0,1])$ has a natural metric:

$$d(f,g) = sup_{x\in X}|f(x) - g(x)|$$

This gives a complete metric space.

We have an endofunction

$$\Phi : Hom(X,[0,1]) \to Hom(X,[0,1])$$

given by “going around the square”:

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ [0,1] \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\langle r,next\rangle} & [0,1-\delta]\times X \\ {\scriptstyle \Phi(f)}\downarrow & & \downarrow{\scriptstyle [0,1]\times f} \\ [0,1] & \xleftarrow[a]{} & [0,1-\delta]\times[0,1] \end{array}$$

One checks that Φ is a contracting map, due to $\delta < 1$.

Then Φ has a unique fixed point,
and any fixed point is a solution to $\langle r, \mathrm{next}\rangle$. □

The operation on sets

$$FX = [0, 1 - \delta] \times X$$

is a functor: for $f : X \to Y$, we have

$$Ff : [0, 1 - \delta] \times X \to [0, 1 - \delta] \times Y$$

given by $Ff(r, x) = (r, f(x))$.

Now our system $\langle r, \text{next}\rangle : X \to FX$ is a coalgebra for F.

And the important map $a : [0, 1 - \delta] \times [0, 1] \to [0, 1]$ is

$$a : F[0, 1] \to [0, 1] \qquad a(x, y) = x + \delta \cdot y$$

is an algebra for F.

Let $\mathcal{A}$ be a category, and let $F : \mathcal{A} \to \mathcal{A}$ be a functor.

An F-algebra is a morphism of the form $a : FA \to A$.
An F-coalgebra is a morphism of the form $a : A \to FA$.

Example: deterministic automata

$$(S, s : S \to 2 \times S^A)$$

are coalgebras of $2 \times X^A$, again on Set.

Let $(A, a : FA \to A)$ and $(B, b : FB \to B)$ be algebras.
A morphism is $f : A \to B$ in the same underlying category so that

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \xrightarrow[b]{} & B \end{array}$$

commutes.

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commutes.

Initial algebras and final coalgebras

initial algebra

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \xrightarrow[b]{} & B \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ B & \xrightarrow[b]{} & FB \end{array}$$

final coalgebra

The category is Set.

The functor is $FX = 1 + X$.

An algebra for F is a set A together with a map

$$1 + A \to A$$

So it is an element $a \in A$ and an endo-map $f : A \to A$.

The main example is $N = \omega$, the natural numbers,
with $0 \in N$, and $s \colon N \to N$ the successor function.

We put s and 0 together to get $[0, s] : 1 + N \to N$.

Recursion on N is tantamount to Initiality

Recursion on N: For all sets A, all $a \in A$, and all $f : A \to A$, there is a unique $\varphi : N \to A$ so that

$$\begin{array}{lcll} \varphi(0) & = & a & \\ \varphi(n+1) & = & f(\varphi(n)) & \text{for all } n \end{array}$$

Initiality of N: For all $(A, [a, f])$, there is a unique homomorphism

$$\varphi : (N, [0, s]) \to (A, [a, f])$$

That is, the diagram below commutes:

$$\begin{array}{ccc} 1 + N & \xrightarrow{[0,s]} & N \\ {\scriptstyle 1+\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ 1 + A & \xrightarrow[{[a,f]}]{} & A \end{array}$$

Recursion on N may be recast in terms of maps out of an initial algebra.

Example: the finite binary trees

The category is Set.

The functor is $FX = 1 + (X \times X)$.

An algebra for F is a set A together with a map

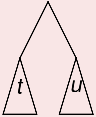
$$[w, a] : 1 + (A \times A) \to A$$

So it is an element $w \in A$ and a map $a : A \times A \to A$.

RECURSION PRINCIPLE FOR FINITE BINARY TREES

For all sets A, all $w \in A$, all $a : A \times A \to A$,
there is a unique $\varphi : \mathit{Trees} \to A$
so that φ is

one-point tree $\bullet \quad \mapsto \quad w$



$\mapsto \quad a(\varphi(t), \varphi(u))$

RECURSION PRINCIPLE FOR FINITE BINARY TREES

For all algebras $[w, a] : 1 + (A \times A) \to A$,
there is a unique $\varphi : \mathit{Trees} \to A$ so that

$$\begin{array}{ccc} 1 + (\mathit{Trees} \times \mathit{Trees}) & \xrightarrow{?} & \mathit{Trees} \\ \Big\downarrow{\scriptstyle 1+(\varphi\times\varphi)} & & \Big\downarrow{\scriptstyle \varphi} \\ 1 + (A \times A) & \xrightarrow[{[w,a]}]{} & A \end{array}$$

commutes, where $(\varphi \times \varphi)(t, u) = (\varphi(t), \varphi(u))$.

functor	initial algebra
$1 + X$ on *Set*	natural numbers
$1 + X^2$ on *Set*	finite binary trees
$1 + (A \times X)$ on *Set*	finite sequences from A
$1 + X^2$ on *MS*	finite binary trees, with metric
$1 + X^2$ on *CMS*	finite and infinite binary trees, with metric

functor	final coalgebra
$1 + X$ on *Set*	natural numbers + ∞
$1 + X^2$ on *Set*	finite and infinite binary trees
$1 + (A \times X)$ on *Set*	finite and infinite sequences from A

In all these cases, the structure maps are also natural.

Example: formal languages as a final coalgebra

The category is *Set*.

The functor is $F(S) = 2 \times S^A$, where A is a fixed “alphabet” set.

Coalgebras of $2 \times X^A$ are deterministic automata

$$
\begin{array}{ccc}
S & \xrightarrow{\;s\;} & 2 \times S^A \\
{\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle id_2 \times \varphi^A} \\
\mathcal{L} & \xrightarrow[\;l\;]{} & 2 \times \mathcal{L}^A
\end{array}
$$

final coalgebra

Let $\mathcal{L} = \mathcal{P}(A^*)$ be the set of formal languages over A

The final coalgebra is

$$\mathcal{L} \to 2 \times \mathcal{L}^A,$$

and is given in terms of Brzozowski derivatives.

The map φ takes a state to the language accepted there.

A lot of interesting structures in discrete mathematics, starting with the set of natural numbers itself, are either initial algebras or final coalgebras.

What about continuous mathematics?

What about the set $\mathbb{R}$ of reals?
What about $[0, 1]$?

This talk is mainly a progress report on work in this area.

Early references:
"Calculus in Coinductive Form"
D. Pavlović; M.H. Escardo, 1998

"On coalgebra of real numbers"
D. Pavlović, V. Pratt, 1999

Fix an endofunctor $F : \mathcal{A} \to \mathcal{A}$ of some category.

An algebra $a : FA \to A$ is corecursive
if for every coalgebra $b : B \to FB$
there is a unique coalgebra-to-algebra morphism $b^\dagger : B \to A$:

$$\begin{array}{ccc} B & \xrightarrow{\;b\;} & FB \\ {\scriptstyle b^\dagger}\downarrow & & \downarrow{\scriptstyle Fb^\dagger} \\ A & \xleftarrow[\;a\;]{} & FA \end{array}$$

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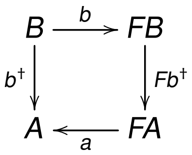

EXAMPLE

For the functor $FX = [0,1] \times X$, the algebra

$$a : F[0,1] \to [0,1] \qquad a(x,y) = x + \delta \cdot y$$

is corecursive, provided $0 \leq \delta < 1$.

Let BiP be the category of bi-pointed sets.

These are triples $(X, \top, \bot)$ with X a set and also $\top, \bot \in X$ and $\top \neq \bot$.

The bipointed set $\{\top, \bot\}$ is initial, but there is no final object.

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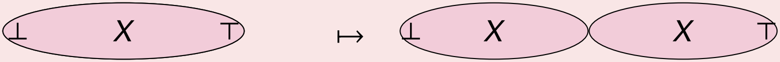
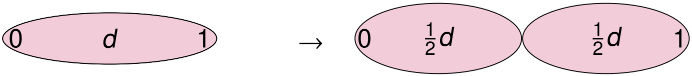
THE FUNCTOR $F : \text{BiP} \to \text{BiP}$



identify $\top$ of left with $\bot$ of right

Let's move from bipointed sets to bipointed metric spaces.

Let $i : [0,1] \to F[0,1] = [0,1]$ be the map



$$
\begin{array}{lcl}
a < \frac{1}{2} & \mapsto & 2a \text{ on left} \\
\frac{1}{2} & \mapsto & \text{midpoint} \\
a > \frac{1}{2} & \mapsto & 2a \text{ on right}
\end{array}
$$

Note that i is an isometry.

Let $X \to FX$ be a coalgebra.

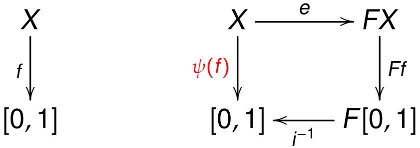
The space

$$S = hom_{\mathrm{BiP}}(X, [0,1]).$$

is a closed subspace of $hom_{\mathrm{CMS}}(X, [0,1])$, hence is complete.

$$d(Ff, Fg) \leq \frac{1}{2} d(f, g).$$

We have a contracting endofunction $\psi : S \to S$:
take $f : X \to [0,1]$ to $i^{-1} \cdot Ff \cdot e$:



By the Contraction Mapping Thm., there's a unique $f^* = \psi(f^*)$.

f^* is exactly a coalgebra morphism $(X, e) \to ([0,1], i)$.

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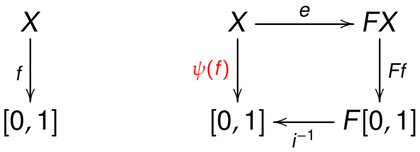

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f^* is exactly a coalgebra morphism $(X, e) \to ([0,1], i)$.

EXERCISE

I'm cheating. But how?

More on $F\colon \mathsf{BiP} \to \mathsf{BiP}$

- On BiP, the initial algebra is the dyadic rationals in $[0, 1]$.
- On BiP, the final coalgebra is the unit interval as a set.
- On bipointed metric spaces, the final coalgebra is the unit interval with the usual metric.

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The final coalgebra turned out to be the Cauchy completion of the initial algebra.

Development: fractals as final coalgebras

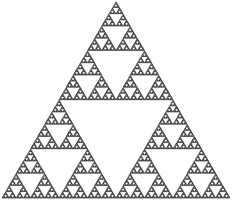
Building on Hutchinson's iterated function systems, fractal subsets of $\mathbb{R}^n$ are often described as final coalgebras.

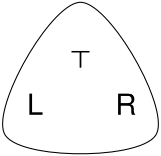
In some (many?) cases those final coalgebras are completions of the initial algebras.

This has been worked out in a few concrete settings:

- the Sierpinski triangle and the circle(!) (with Prasit Bhattacharya, Jayampathy Ratnayake, and Robert Rose)
- the Sierpinski carpet, including complex gluing. (with Victoria Noquez)

The Sierpiński Gasket as a Final Coalgebra



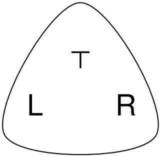


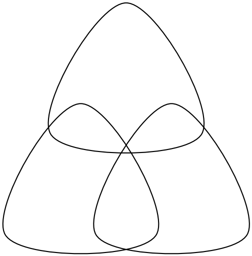
A tripointed set is a set X together with distinguished different elements $\top$, L, and R.

Morphisms are functions preserving $\top$, L, and R.

The initial object I of Tri is $\{\top, L, R\}$.
But Tri has no final object.

Here is a generic tripointed set:





The functor F takes this to 3 copies with identifications as shown above. In a tripointed metric space:

- all 3 distinguished points have distance 1
- the functor squashes distances by $1/2$

category	initial algebra	final coalgebra
Set_3 Tripointed sets	(G, g) "finite address space" of the gasket $\mathbb{S}$	its completion (S, s) also $(\mathbb{S}, \sigma) =$ the Sierpinski Gasket as a subset of $\mathbb{R}^2$
$\mathrm{Met}_3{}^{Sh}$ short maps	(G, g)	(S, s)
$\mathrm{Met}_3{}^{L}$ Lipschitz maps	(G_ρ, g) G with discrete metric	none exists
$\mathrm{Met}_3{}^{C}$ continuous maps	(G_ρ, g)	(S, s) and $(\mathbb{S}, \sigma)$ they are bilipschitz isomorphic

For $FX = N \times X$

$$N \times [0,1] \xrightarrow{(n,r)\mapsto \frac{n+r}{1+n+r}} [0,1]$$

$$N \times \mathbb{R}^{\geq 0} \xrightarrow{(n,r)\mapsto n+\frac{r}{1+r}} \mathbb{R}^{\geq 0}$$

Concerning $\mathbb{R}^{\geq 0}$

Given $e : X \to N \times X$, we want a unique $e^{\dagger}$:

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & N \times X \\ {\scriptstyle e^{\dagger}} \downarrow & & \downarrow {\scriptstyle N \times e^{\dagger}} \\ \mathbb{R}^{\geq 0} & \xleftarrow[s(n,r) = n + \frac{r}{1+r}]{} & N \times \mathbb{R}^{\geq 0} \end{array}$$

Let's adopt notation for $f : X \to N \times X$:

$f(x_0) = (a_0, x_1) \qquad f(x_1) = (a_1, x_2) \qquad \cdots \qquad f(x_n) = (a_n, x_{n+1}) \qquad \cdots$

Then we are asking if we can solve the system

$$\begin{aligned} x_0 &= a_0 + \frac{1}{1+x_1} \\ x_1 &= a_1 + \frac{1}{1+x_2} \\ &\vdots \\ x_n &= a_n + \frac{1}{1+x_{n+1}} \\ &\vdots \end{aligned}$$

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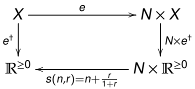

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$$x_0 = a_0 + \cfrac{1}{1 + a_1 + \frac{1}{1+a_2+\cdots}}$$

The theory of continued fractions implies that we have a corecursive algebra.

This point is due to Dusko Pavlović and Vaugh Pratt in their 1999 paper "On coalgebra of real numbers"

What is the sum below?

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}$$

What is the sum below?

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}$$

We are solving the system (=coalgebra)

$$x = \frac{1}{1+x}$$

and so we get

$$x^{\dagger} = \frac{-1+\sqrt{5}}{2}$$

More sophisticated example of a proof in this area, due to Jayampathy Ratnayake 2022

Theorem (Known)

For every sequence $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \ldots)$ *of digits* 1 *or* −1,

$$\varepsilon_0 \sqrt{2 + \varepsilon_1 \sqrt{2 + \varepsilon_2 \sqrt{2 + \cdots}}} = 2 \sin\left(\frac{\pi}{2}\left(\frac{\varepsilon_0}{2} + \frac{\varepsilon_0 \varepsilon_1}{4} + \frac{\varepsilon_0 \varepsilon_1 \varepsilon_2}{8} + \cdots\right)\right).$$

For example,

$$\sqrt{2 - \sqrt{2 + \sqrt{2 - \cdots}}} = \frac{1 + \sqrt{5}}{2}$$

The point

A proof using final coalgebras and/or corecursive algebras is arguably easier than the classical proof.

Working out this kind of thing should teach us quite a bit.

Consider the functor $FX = \{-1, 1\} \times X$.

$$\begin{array}{ccc}
\{-1,1\} \times [-2,2] & \xrightarrow{g(\varepsilon,x)=\varepsilon\sqrt{2+x}} & [-2,2] \\
\Big\downarrow{\scriptstyle \{-1,1\}\times i} & & \Big\uparrow{\scriptstyle i^{-1}(x)=2\sin(\frac{\pi}{2}x)} \\
\{-1,1\} \times [-1,1] & \xrightarrow[f(\varepsilon,x)=\frac{\varepsilon}{2}(1+x)]{} & [-1,1]
\end{array}$$

Both horizontal maps are corecursive algebra structures.

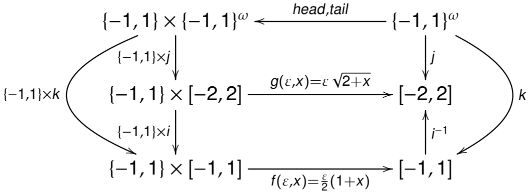
The vertical map

$$i(x) = \frac{2}{\pi} \arcsin \frac{x}{2}$$

is an isomorphism of F-algebras, and

$$i^{-1}(x) = 2\sin(\frac{\pi}{2}x)$$

Proof, based on work of Jayampathy Ratnayake



The maps j and k are coalgebra-to-algebra maps from the streams into the two corecursive algebras.

Then j satisfies

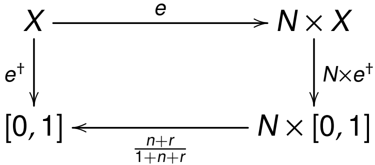
$$\begin{array}{rcl} j(\epsilon_0,(\epsilon_1,\epsilon_2,\ldots)) & = & \epsilon_0\sqrt{2+j(\epsilon_1,\epsilon_2,\ldots)} \\ & = & \epsilon_0\sqrt{2+\epsilon_1\sqrt{2+j(\epsilon_2,\epsilon_3,\ldots)}} \end{array}$$

And $k : \{1,-1\}^\infty \to \{1,-1\}$ satisfies

$$\begin{array}{rcl} k(\epsilon_0,(\epsilon_1,\epsilon_2,\ldots)) & = & \frac{\epsilon_0}{2}(1+k(\epsilon_1,\epsilon_2,\ldots)) \\ & = & \frac{\epsilon_0}{2}(1+\frac{\epsilon_1}{2}(1+k(\epsilon_2,\epsilon_3,\ldots))) \end{array}$$

And we also have $i^{-1} \cdot k = j$.
This basically gives us the identity we are after.

Given $e : X \to N \times X$, we want a unique $e^\dagger$:



Given $e : X \to N \times X$, we want a unique $e^\dagger$:

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & N \times X \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle N\times e^\dagger} \\ [0,1] & \xleftarrow[\frac{n+r}{1+n+r}]{\quad} & N \times [0,1] \end{array}$$

We can’t use the Contracting Mapping Theorem,
and so more work is needed.

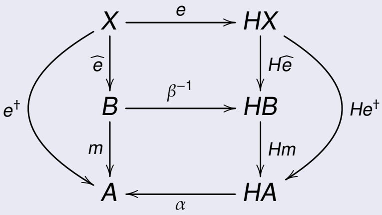
Noquez and I used linear fractional transformations.

Lemma (Adámek, Milius, LM)

Let H be any endofunctor, let (A, α) be a corecursive H-algebra.

Let (B, β) be a fixed point of H which is a subalgebra of A.

Assume that for every coalgebra $e : X \to HX$, the coalgebra-to-algebra map $e^\dagger$ factors through the algebra morphism $m : B \to A$.



Then (B, β^{-1}) is the final coalgebra of H.

	$FX = N \times X$	$GX = N \times X + 1$
corecursive algebra:		
carrier	$\mathbb{R}^{\geq 0} =$ reals ≥ 0	$\mathbb{R}^{\geq 0}$
structure	$\alpha(n, r) = n + \frac{1}{1+r}$	$[\alpha, 0]$
final coalgebra		
carrier	$\mathcal{A} =$ irrationals > 0	$\mathbb{R}^{\geq 0}$
inverse structure	$\alpha_0 =$ restriction of α to $\mathcal{A}$	$[\alpha, 0]$
structure	$\gamma(x) = (\lfloor x \rfloor, \frac{1}{x \bmod 1} - 1)$	$\chi(0) \in 1$ in $G(\mathbb{R}^{\geq 0})$ $\chi(x) = (x - 1, 0)$ for $x \geq 1$ in N else $\chi(x) = (\lfloor x \rfloor, \frac{1}{x \bmod 1} - 1)$
final coalgebra isomorphic copy:		
carrier	$\mathcal{B} =$ irrationals $\cap [0, 1]$	$(0, 1]$
inverse structure	$\beta_0 =$ restriction of β to $\mathcal{B}$	$[\beta, 1]$, where $\beta(n, r) = 1/(1 + n + r)$
structure	$\delta(x) = (\left\lfloor \frac{1}{x} \right\rfloor - 1, \frac{1}{x} \bmod 1)$	$\rho(1) \in 1$ in $G((0, 1])$ $\rho(x) = ((1/x) - 2, 1)$ if $1/x \in N \setminus \{1\}$ else $\rho(x) = (\left\lfloor \frac{1}{x} \right\rfloor - 1, \frac{1}{x} \bmod 1)$

More corecursive algebras and final coalgebra structures

	$FX = N \times X$
corecursive algebra:	
carrier	$\mathbb{I} = [0, 1]$
structure	$\tau(n, r) = \frac{n+r}{1+n+r}$
final coalgebra	
carrier	$[0, 1)$
inverse structure	$\sigma(n, r) = \frac{n+r}{1+n+r}$
structure	$\zeta(x) = (\lfloor \frac{x}{1-x} \rfloor, \frac{x}{1-x} \bmod 1)$
final coalgebra isomorphic copy:	
carrier	$\mathbb{R}^{\geq 0}$
inverse structure	$\theta(n, r) = n + \frac{r}{1+r}$
structure	$\xi(x) = (\lfloor x \rfloor, \frac{x \bmod 1}{1-x \bmod 1})$

Adámek 1974

Assume that $\mathcal{A}$ has a colimit L of the initial-algebra sequence

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 !} \cdots \qquad \xrightarrow{F^{n-1}!} F^n 0 \xrightarrow{F^n !} \cdots$$

There is a canonical morphism $m\colon L \to FL$.
Assume also that $F : \mathcal{A} \to \mathcal{A}$ preserves the colimit above.
Then

$$(L, m^{-1})$$

is an initial F-algebra.

The "go to" result for initial algebras

Kleene's Theorem

Let $\mathcal{A} = (A, \leq)$ be a poset, and $F\colon \mathcal{A} \to \mathcal{A}$ be monotone.
Assume that $\mathcal{A}$ has a least upper bound L of the a sequence

$$0 \leq F0 \leq F^2 0 \leq \cdots \leq F^n 0 \leq \cdots$$

Then $L \leq FL$.
Assume also that $F : \mathcal{A} \to \mathcal{A}$ is continuous
or at least preserves the least upper bound above.
Then L is the least fixed point of F.

Adámek 1974

Assume that $\mathcal{A}$ has a colimit L of the initial-algebra sequence

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 !} \cdots \qquad \xrightarrow{F^{n-1}!} F^n 0 \xrightarrow{F^n !} \cdots$$

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ADÁMEK 1974

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There is a canonical morphism $m\colon L \to FL$.
Assume also that $F : \mathcal{A} \to \mathcal{A}$ preserves the colimit above.
Then

$$(L, m^{-1})$$

is an initial F-algebra.

BARR 1993

Assume that $\mathcal{A}$ has a limit L of the final-coalgebra sequence

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} \qquad \cdots \xleftarrow{F^{n-1}!} F^n 1 \xleftarrow{F^n!} \cdots$$

There is a canonical morphism $m\colon FL \to L$.
Assume also that $F : \mathcal{A} \to \mathcal{A}$ preserves the limit above.
Then

$$(L, m^{-1})$$

is an final F-coalgebra.

Getting a final coalgebra a la Barr only works
for a limited class of functors,
leaving out some which come up often.

The first example where this fails

$\mathcal{P}_{fin}X =$ the set of finite subsets of X.
For $g\colon X \to Y$,

$$\mathcal{P}_{fin}g(A) = g[A]$$

for all finite subsets $A \subseteq X$.

$\mathcal{P}_{fin}$ has a final coalgebra, but it's not L.

The functors below all have final coalgebras

Kripke polynomial functors on Set: adaptation of Worrell 1995

$$F ::= \mathcal{P}_{fin} \mid A \mid Id \mid \prod_{i \in I} F_i \mid \coprod_{i \in I} F_i \mid FF,$$

where A ranges over constant functors for sets.
and I is an arbitrary index set.
$\mathcal{P}_{fin}$ is the finite power set functor.

Hausdorff polynomial functors on Met (AMM 2023)

$$F ::= H \mid A \mid Id \mid \prod_{i \in I} F_i \mid \coprod_{i \in I} F_i \mid FF,$$

H is the Hausdorff functor.

Vietoris polynomial functors on Haus (AMM 2023)

$$F ::= V \mid A \mid Id \mid \prod_{i \in I} F_i \mid \coprod_{i \in I} F_i \mid FF,$$

V is the Vietoris functor.

The Hausdorff functor $H\colon \mathrm{Met} \to \mathrm{Met}$

Met is the category of extended metric spaces
(allowing $d(x,y) = \infty$),
and non-expanding maps.

The Hausdorff functor

$$H\colon \mathrm{Met} \to \mathrm{Met}$$

maps a metric space X
to the space HX of all compact subsets of X
equipped with the Hausdorff distance:

$$\bar{d}(S,T) = \max\left(\sup_{x\in S} d(x,T), \sup_{y\in T} d(y,S)\right),$$

where $d(x,S) = \inf_{y\in S} d(x,y)$.

In particular $\bar{d}(\emptyset, T) = \infty$ for nonempty compact sets T.

For a non-expanding map $f\colon X \to Y$ we take $Hf\colon S \mapsto f[S]$.

Let X be a topological space.

VX is the space of compact subsets of X
equipped with the "hit-and-miss" topology.

This topology has as a subbase all sets of the following forms:

$$U^{\Diamond} = \{R \in VX : R \cap U \neq \emptyset\} \qquad (R \text{ hits } U),$$
$$U^{\Box} = \{R \in VX : R \subseteq U\} \qquad (R \text{ misses } X \setminus U),$$

where U ranges over the open sets of X.

VX is the Vietoris space of X,
also known as the hyperspace of X.

In Set and Met, we get final coalgebras by taking the limit L of

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} \quad \cdots \xleftarrow{F^{n-1}!} F^n 1 \xleftarrow{F^n!} \cdots$$

and taking a second infinite limit

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} \quad \cdots \quad L \xleftarrow{m} FL \xleftarrow{Fm} F^2 L \cdots$$

In this second iteration all of the morphisms are monomorphisms,

The functors which I mentioned do preserve limits where the morphisms are monic.

Logicians are very familiar with recursion and induction, perhaps less so with their "duals" corecursion and coinduction.

Many of the fundamental structures in mathematical logic happen to be initial algebras:
the natural numbers, or the cumulative hierarchy of sets.

At the same time, there are many compelling structures in continuous math
that are characterized as final coalgebras:
the Cantor space, the unit interval, fractals, and Harsanyi type spaces.

This talk was a high-level introduction to the area of coalgebra, tuned to a logic audience.

Part of the appeal of coalgebra in theoretical computer science is that it gives a set of tools relevant and applicable to finitely approximable infinite objects.

These same tools can be pointed back at more "classical" topics, like those in areas of continuous mathematics.

This talk is a kind of progress report on this turn.

It has been more like an examination of special topics and less of a general theory.

- On initial algebras and terminal coalgebras:
  Jiří Adámek, Stefan Milius

- On the Sierpinski triangle:
  Prasit Bhattacharya, Jayampathy Ratnayake, Robert Rose

  Jayampathy Ratnayake, Annanthakrishna Manokaran,
  Romaine Jayewarden, Victoria Noquez

- On the Sierpinski carpet:
  Victoria Noquez

- On fractals and process calculi:
  Todd Schmid, Victoria Noquez