A j-translation with Kripke forcing relation

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LLAMA seminar
ILLC, the University of Amsterdam, February 11, 2026

Key Points of This Talk

- The j-translation have found many applications in proof theory. It also appears in topos theory and realizability theory.

$$\varphi \quad \mapsto \quad \varphi^{j}$$

- De Jongh and Goodman introduced a realizability with forcing. This has led to various applications.

$$f \Vdash_{T} n \, \mathbf{r} \, \varphi$$

- Our main purpose is to provide a proof-theoretic counterpart of de Jongh-Goodman realizability from the perspective of j-translation.

$$\varphi \quad \mapsto \quad j \Vdash_{\mathbb{P}} \varphi$$

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Introduction: j-translation in various contexts

j-translation in $\mathbf{IHoL}$

j-translation with Kripke forcing

The corresponding realizability semantics

j-translation in Proof theory

- In intuitionistic proof theory, various syntactic translations have provided many insights and applications for decades.
- A typical example is the **negative translation** (or **double negation translation**).

$$
\begin{array}{ll}
(R[\vec{x}])^{\mathrm{N}} := \neg\neg R[\vec{x}]; & (\varphi \to \psi)^{\mathrm{N}} := \varphi^{\mathrm{N}} \to \psi^{\mathrm{N}}; \\
(\varphi \wedge \psi)^{\mathrm{N}} := \varphi^{\mathrm{N}} \wedge \psi^{\mathrm{N}}; & (\varphi \vee \psi)^{\mathrm{N}} := \neg\neg(\varphi^{\mathrm{N}} \vee \psi^{\mathrm{N}}); \\
(\exists x.\varphi)^{\mathrm{N}} := \neg\neg(\exists x \varphi^{\mathrm{N}}); & (\forall x.\varphi)^{\mathrm{N}} := \forall x \varphi^{\mathrm{N}}.
\end{array}
$$

It is well known that this translation defines a uniform way to embed classical logic $\mathbf{CQC}$ into intuitionistic logic $\mathbf{IQC}$.

Proposition

For any first-order formula φ, $\vdash_{\mathbf{CQC}} \varphi \iff \vdash_{\mathbf{IQC}} \varphi^{\mathrm{N}}$.

- This translation is known as an example of the j-translation associated with a nucleus j.

Definition (Nucleus)

A function $j\colon \mathrm{Fml} \to \mathrm{Fml}$ on the set of formulas is called a **nucleus** if the following implications are intuitionistically provable:

$$\vdash_{\mathbf{IQC}} \varphi \to j\varphi; \qquad \vdash_{\mathbf{IQC}} j(j\varphi) \to j\varphi;$$
$$\vdash_{\mathbf{IQC}} (\varphi \to \psi) \to (j\varphi \to j\psi); \quad \vdash_{\mathbf{IQC}} (j\varphi)[t/x] \leftrightarrow j(\varphi[t/x]).$$

Definition (Gödel-Gentzen-style j-translation)

Given a nucleus j, the j-**translation** $\varphi^j[\vec{x}]$ of $\varphi[\vec{x}]$ is defined as follows:

$$(R[\vec{x}])^j :\equiv jR[\vec{x}]; \qquad (\varphi \to \psi)^j :\equiv \varphi^j \to \psi^j;$$
$$(\varphi \wedge \psi)^j :\equiv \varphi^j \wedge \psi^j; \quad (\varphi \vee \psi)^j :\equiv j(\varphi^j \vee \psi^j);$$
$$(\exists x.\varphi)^j :\equiv j(\exists x \varphi^j); \quad (\forall x.\varphi)^j :\equiv \forall x \varphi^j.$$

Various j-translations have found practical applications, such as **relative consistency** and **partial conservation** results.

- $j\varphi :\equiv \neg\neg\varphi$ (negative translation)
- $j\varphi :\equiv (\varphi \to A) \to A$ (A-negative translation)
- $j\varphi :\equiv (\varphi \vee A)$ (Friedman translation)
- $j\varphi :\equiv (\varphi \to A) \to \varphi$ (Peirce translation)

For j-translations, the following properties are fundamental.

Lemma

For any nucleus j and any formula φ,

1. $\vdash_{\mathbf{IQC}} (j\varphi^j \leftrightarrow \varphi^j)$. ($j$-closedness)
2. $\vdash_{\mathbf{IQC}} (\mathbf{IQC})^j$. (Soundness for $\mathbf{IQC}$)

Remark: By taking $j = \neg\neg$, we obtain the property that the negative translation embeds $\mathbf{CQC}$ into $\mathbf{IQC}$ as a corollary.

j-translation in Topos theory

- The notions of a nucleus and the associated j-translation appear naturally in topos theory. A topos $\mathcal{E}$ is equipped with a subobject classifier Ω. It can interpret various mathematical propositions. (Topos as a universe of mathematics)
- Lawvere and Tierney investigated logical aspects of topos theory. They showed the following correspondence:

$$\text{a nucleus } j \text{ on } \Omega \quad \overset{1:1}{\longleftrightarrow} \quad \text{a subtopos } \mathcal{E}_j \subseteq \mathcal{E}.$$

In this context, such a nucleus is called a **local operator**.

Proof theory	Topos theory
Fml	subobject classifier Ω
nucleus j	local operator j
j-translation	validity in $\mathcal{E}_j \subseteq \mathcal{E}$
$\vdash j\varphi \to k\varphi$	$\mathcal{E}_k \subseteq \mathcal{E}_j$

j-translation in Realizability theory

- Hyland's discovery of the **effective topos** $\mathcal{E}ff$ connects the j-translation with realizability theory. The original realizability notion, **Kleene realizability**, is based on Turing computability. The validity in $\mathcal{E}ff$ coincides with Kleene realizability. ($\mathcal{E}ff$ as a universe of computable mathematics)
- For a partial function f on $\mathbb{N}$, there is a local operator j_f such that:

$$\mathcal{E}ff_{j_f} \models \varphi \iff \varphi \text{ is Kleene realizable relative to } f.$$

In this sense, a local operator is regarded as a generalized oracle.

Proof theory	Topos theory	Realizability theory
Fml	subobject classifier Ω	$\mathcal{P}(\mathbb{N})$
nucleus j	local operator j	generalized oracle j
j-translation	validity in $\mathcal{E}_j \subseteq \mathcal{E}$	j-relative realizability
$\vdash j\varphi \to k\varphi$	$\mathcal{E}_k \subseteq \mathcal{E}_j$	j is reducible to k

De Jongh-Goodman realizability PF : a set of partial functions on $\mathbb{N}$

$$f \Vdash_T (n \; \mathbf{r} \; \varphi), \quad \text{where} \begin{cases} f \in \mathrm{PF} \text{ is used as an oracle} \\ T \subseteq \mathrm{PF} \text{ is used as a forcing poset} \end{cases}$$

- De Jongh and Goodman independently introduced a **sheaf model of realizability** to prove the following conservation results [de Jongh 69], [Goodman 78].

Theorem (De Jongh's theorem)

If $\nvdash_{\mathbf{IPC}} \varphi[\vec{p}]$, then there exist $\mathbf{HA}$-formulas $\vec{\sigma}$ such that $\mathbf{HA} \nvdash \varphi[\vec{\sigma}]$.

Theorem (Goodman's theorem)

$\mathbf{HA}^\omega + \mathbf{AC}^\omega$ is conservative over $\mathbf{HA}$.

- Van Oosten pointed out that this variant can be understood as a **PCA-valued sheaf** [van Oosten 91].

De Jongh-Goodman realizability PF : a set of partial functions on $\mathbb{N}$

$$f \Vdash_T (n \mathbin{\mathbf{r}} \varphi), \quad \text{where } \begin{cases} f \in \mathrm{PF} \text{ is used as an oracle} \\ T \subseteq \mathrm{PF} \text{ is used as a forcing poset} \end{cases}$$

Question

What is a proof-theoretic (or topos-theoretic) counterpart of de Jongh-Goodman realizability?

Proof theory	Topos theory	Realizability theory
Fml	subobject classifier Ω	$\mathcal{P}(\mathbb{N})$
nucleus j	local operator j	generalized oracle j
j-translation	validity in $\mathcal{E}_j \subseteq \mathcal{E}$	j-relative realizability
$\vdash j\varphi \to k\varphi$	$\mathcal{E}_k \subseteq \mathcal{E}_j$	j is reducible to k
	$j \Vdash_{\mathbb{P}} \varphi$	$f \Vdash_T (n \mathbin{\mathbf{r}} \varphi)$

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Our Objective

To provide a syntactic treatment of a nucleus $j \in \mathrm{Fml}^{\mathrm{Fml}}$ and a family $\mathbb{P} \subseteq \mathrm{Fml}^{\mathrm{Fml}}$ of nuclei.

For this purpose, the internal logic of a topos, **Intuitionistic Higher-order Logic (IHoL)**, is suitable.

$$\begin{cases} p \in \mathrm{Fml} \\ j \in \mathrm{Fml}^{\mathrm{Fml}} \\ \mathbb{P} \in \mathcal{P}(\mathrm{Fml}^{\mathrm{Fml}}) \end{cases} \quad \rightsquigarrow \quad \begin{cases} p : \Omega \\ j : P\Omega \\ \mathbb{P} : P(P\Omega) \end{cases} \quad \text{in } \mathbf{IHoL}$$

To explain this, let us briefly recall the basics of **IHoL**.

Definition (Elementary topos)

A category $\mathcal{E}$ is an **(elementary) topos** if it has:

- a terminal object 1,
- a subobject classifier $(\Omega, \top\colon 1 \to \Omega)$,
- binary products $X \times Y$ for all X, $Y \in \mathcal{E}$,
- power objects $(PX, \in_X\colon X \times PX \to \Omega)$ for all $X \in \mathcal{E}$.

Definition (Internal language of a topos)

The **internal language** $\mathcal{L}_\mathcal{E}$ of a topos $\mathcal{E}$ consists of:

Sorts $X, Y ::= A \in \mathcal{E} \mid 1 \mid \Omega \mid X \times Y \mid PX$,

Terms $t, s ::= x : X \mid * : 1 \mid \top : \Omega \mid \langle t, s\rangle : X \times Y \mid (t =_X s) : \Omega \mid (t \in_X s) : \Omega \mid \{ x : X \mid \varphi \} : PX$,

where φ denotes a term of type Ω.
Every term of type Ω is called **($\mathcal{L}_\mathcal{E}$-)formulas**.

Fact

- Logical connectives $\bot$, $\wedge$, $\vee$, $\to$ and quantifiers $\exists x : X$, $\forall x : X$ are definable. For instance, universal quantification are:

$$\forall x : X.\varphi[x] := (\{\, x : X \mid \varphi[x] \,\} =_{PX} \{\, x : X \mid \top \,\})$$

- For any $\mathcal{L}_{\mathcal{E}}$-formula $\varphi : \Omega$, the validity $\mathcal{E} \models \varphi$ is defined.
The corresponding logic is called **Intuitionistic Higher-order Logic ($\mathbf{IHoL}$)**:

$$\mathcal{E} \models \mathbf{IQC}.$$
$$\mathcal{E} \models \forall y : X.((y \in_X \{\, x : X \mid \varphi[x] \,\}) \leftrightarrow \varphi[y]).$$

- $\mathbf{IHoL}$ is often referred to as **local set theory**. However, there is a major restriction compared to intuitionistic set theory:

✓ <u>bounded quantification</u> : for all x of type $X, \cdots$

× <u>unbounded quantification</u> : for all object (set), $\cdots$

- In **IHoL**, the power object PX is isomorphic to the exponential Ω^X. In particular, an endomorphism $j\colon \Omega \to \Omega$ can be treated as a term of type $P\Omega$.
- Therefore, we can use quantification over local operators.

Definition (Local operator, internally)

A formula $\text{is-lop}[j]$ with a free variable $j : P\Omega$ is defined by:

$$\begin{aligned}\text{is-lop}[j] :=& \forall p\colon \Omega(p \to jp) \land \forall p : \Omega.(j(jp) \to jp)\\ &\land \forall p, q : \Omega.((p \to q) \to (jp \to jq)).\end{aligned}$$

We then define $\forall j \in \text{Lop}.\varphi[j] := \forall j : P\Omega.(\text{is-lop}[j] \to \varphi[j])$.

For simplicity, we restrict our attention to one-sorted first-order formulas.

Definition ($\mathcal{L}_X$-formula)

For an object $X \in \mathcal{E}$, $\mathcal{L}_X$-**formulas** are defined by:

$$\varphi, \psi ::= R[\vec{x}] \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \to \psi \mid \exists x : X.\varphi \mid \forall x : X.\varphi.$$

Definition (Internal j-translation in $\mathbf{IHoL}$)

Let $\mathcal{E}$ be an elementary topos and $X \in \mathcal{E}$. For any $\mathcal{L}_X$-formula $\varphi[\vec{x}]$, we inductively define $\varphi^*[j, \vec{x}]$ as follows:

$$(R[\vec{x}])^* := jR[\vec{x}]; \qquad (\varphi \to \psi)^* := \varphi^* \to \psi^*;$$

$$(\varphi \wedge \psi)^* := \varphi^* \wedge \psi^*; \quad (\varphi \vee \psi)^* := j(\varphi^* \vee \psi^*);$$

$$(\exists y : X.\varphi[\vec{x}, y])^* := j(\exists y : X.\varphi^*[j, \vec{x}, y]);$$

$$(\forall y : X.\varphi[\vec{x}, y])^* := \forall y : X.\varphi^*[j, \vec{x}, y].$$

$$\varphi \quad [\vec{x} : X] \quad \mapsto \quad \varphi^* \quad [j : P\Omega, \vec{x} : X].$$

For simplicity, we write $\varphi^j[\vec{x}] := \varphi^*[j, \vec{x}]$.

Lemma

For any $\mathcal{L}_X$-formula φ,

1. $\mathcal{E} \models \forall j \in \mathrm{Lop}.(j\varphi^j \leftrightarrow \varphi^j)$. ($j$-closedness)
2. $\mathcal{E} \models \forall j \in \mathrm{Lop}.(\mathbf{IQC})^j$. (Soundness for $\mathbf{IQC}$)

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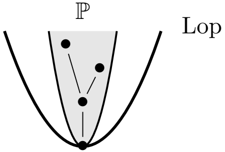
In **IHoL**, the standard order on Lop is defined internally. Furthermore, we can express that "$\mathbb{P}$ is a subset of Lop".

Definition

For terms $j, k : P\Omega$, and $\mathbb{P} : P(P\Omega)$,

- $(j \leq k) \coloneqq \forall p : \Omega.(jp \to kp)$.
  This formula defines an internal poset $(\mathrm{Lop}, \leq)$ in $\mathcal{E}$.
- $(\mathbb{P} \subseteq \mathrm{Lop}) \coloneqq \forall j : P\Omega.(j \in_\Omega \mathbb{P} \to \text{is-lop}[j])$.
  We call $\mathbb{P}$ a **lop-frame** if $\mathcal{E} \models (\mathbb{P} \subseteq \mathrm{Lop})$ holds.
- $(\forall k \geq_{\mathbb{P}} j.\varphi[k]) \coloneqq \forall k : P\Omega.(k \in_{P\Omega} \mathbb{P} \land j \leq k \to \varphi[k])$.

That is, a lop-frame $\mathbb{P}$ is intended to be an internal subposet of Lop:



Definition

Let $\mathcal{E}$ be an elementary topos and $X \in \mathcal{E}$.
For any $\mathcal{L}_X$-formula φ, we inductively define $j \Vdash_{\mathbb{P}} \varphi$ as follows:

$$
\begin{aligned}
j \Vdash_{\mathbb{P}} (R[\vec{x}]) &:= jR[\vec{x}]; \\
j \Vdash_{\mathbb{P}} (\varphi \wedge \psi) &:= (j \Vdash_{\mathbb{P}} \varphi) \wedge (j \Vdash_{\mathbb{P}} \psi); \\
j \Vdash_{\mathbb{P}} (\varphi \vee \psi) &:= j((j \Vdash_{\mathbb{P}} \varphi) \vee (j \Vdash_{\mathbb{P}} \psi)); \\
j \Vdash_{\mathbb{P}} (\varphi \to \psi[\vec{x}]) &:= \\
&\quad \forall k \geq_{\mathbb{P}} j.((k \Vdash_{\mathbb{P}} \varphi[\vec{x}]) \to (k \Vdash_{\mathbb{P}} \psi[\vec{x}])); \\
j \Vdash_{\mathbb{P}} (\exists y : X.\varphi[\vec{x}, y]) &:= j(\exists y : X.j \Vdash_{\mathbb{P}} \varphi[\vec{x}, y]); \\
j \Vdash_{\mathbb{P}} (\forall y : X.\varphi[\vec{x}, y]) &:= \forall k \geq_{\mathbb{P}} j \forall y : X.k \Vdash_{\mathbb{P}} \varphi[\vec{x}, y].
\end{aligned}
$$

$$\varphi : \Omega \quad [\vec{x} : X] \quad \mapsto \quad j \Vdash_{\mathbb{P}} \varphi : \Omega \quad [\mathbb{P} : P(P\Omega), j : P\Omega, \vec{x} : X].$$

$$j \Vdash_{\mathbb{P}} \varphi \quad = \quad j\text{-translation} \quad + \quad \text{Kripke forcing relation on } \mathbb{P}$$

Lemma

For any $\mathcal{L}_X$-formula φ,

1. (j-closedness)

$$\mathcal{E} \models \forall \mathbb{P} \subseteq \mathrm{Lop} \forall j \in \mathrm{Lop}.(j(j \Vdash_{\mathbb{P}} \varphi) \leftrightarrow j \Vdash_{\mathbb{P}} \varphi).$$

2. (Monotonicity)

$$\mathcal{E} \models \forall \mathbb{P} \subseteq \mathrm{Lop} \forall j \in \mathrm{Lop} \forall k \geq_{\mathbb{P}} j.(j \Vdash_{\mathbb{P}} \varphi \to k \Vdash_{\mathbb{P}} \varphi).$$

Theorem (N.)

$$\mathcal{E} \models \forall \mathbb{P} \subseteq \mathrm{Lop} \forall j \in \mathrm{Lop}.j \Vdash_{\mathbb{P}} (\mathbf{IQC}).$$

Natural numbers object and Heyting arithmetic

Assume that $\mathcal{E}$ has a **natural numbers object** $(N, 0, \mathrm{s})$. Then:

- Every $\mathbf{HA}$-formula can be canonically interpreted as an $\mathcal{L}_N$-formula.

$$\forall x.\varphi \quad \mapsto \quad \forall x : N.\varphi.$$

- Under this interpretation, $\mathcal{E} \models \mathbf{HA}$ holds.

Theorem (N.)

If $\mathcal{E}$ has a natural numbers object,

$$\mathcal{E} \models \forall \mathbb{P} \subseteq \mathrm{Lop} \forall j \in \mathrm{Lop}. j \Vdash_{\mathbb{P}} (\mathbf{HA}).$$

Proof (scketch).

Show that the induction axiom scheme is forced:

$$I_\varphi := \varphi[0] \land \forall x : N.(\varphi[x] \to \varphi[\mathrm{s}(x)]) \to \forall x : N.\varphi[x].$$

- Fix $j \in \mathrm{Lop}$. To show $j \Vdash_{\mathbb{P}} I_\varphi$, assume $k \in \mathbb{P}$ and $j \leq k$.
- $k \Vdash_{\mathbb{P}} (\varphi[0])$ is equivalent to $\forall \ell \geq_{\mathbb{P}} k.\ell \Vdash_{\mathbb{P}} \varphi[0]$.
- $k \Vdash_{\mathbb{P}} (\forall x : N.(\varphi[x] \to \varphi[\mathrm{s}(x)]))$ is equivalent to

$$\forall \ell \geq_{\mathbb{P}} k \forall x : N.(\ell \Vdash_{\mathbb{P}} \varphi[x] \to \ell \Vdash_{\mathbb{P}} \varphi[\mathrm{s}(x)]).$$

- Since N is a natural numbers object, the induction for $\ell \Vdash_{\mathbb{P}} \varphi[x]$ holds. Hence, we obtain:

$$\forall k \geq_{\mathbb{P}} j.(k \Vdash_{\mathbb{P}} (\varphi[0]) \land k \Vdash_{\mathbb{P}} (\forall x : N.(\varphi[x] \to \varphi[\mathrm{s}(x)])) \\ \to k \Vdash_{\mathbb{P}} (\forall x : N.\varphi[x])).$$

- This implies $j \Vdash_{\mathbb{P}} I_\varphi$. □

Our translation $j \Vdash_{\mathbb{P}} \varphi$ was based on the Gödel-Gentzen-style. Alternatively, we can define a translation inspired by the **Kuroda-style j-translation** [van den Berg 19]:

$$j \Vdash^{\mathrm{K}}_{\mathbb{P}} (R[\vec{x}]) \coloneqq R[\vec{x}];$$

$$j \Vdash^{\mathrm{K}}_{\mathbb{P}} (\varphi \wedge \psi) \coloneqq (j \Vdash^{\mathrm{K}}_{\mathbb{P}} \varphi) \wedge (j \Vdash^{\mathrm{K}}_{\mathbb{P}} \psi);$$

$$j \Vdash^{\mathrm{K}}_{\mathbb{P}} (\varphi \vee \psi) \coloneqq (j \Vdash^{\mathrm{K}}_{\mathbb{P}} \varphi) \vee (j \Vdash^{\mathrm{K}}_{\mathbb{P}} \psi);$$

$$j \Vdash^{\mathrm{K}}_{\mathbb{P}} (\varphi \to \psi[\vec{x}]) \coloneqq \forall k \geq_{\mathbb{P}} j.((k \Vdash^{\mathrm{K}}_{\mathbb{P}} \varphi[\vec{x}]) \to k(k \Vdash^{\mathrm{K}}_{\mathbb{P}} \psi[\vec{x}]));$$

$$j \Vdash^{\mathrm{K}}_{\mathbb{P}} (\exists y : X.\varphi[\vec{x}, y]) \coloneqq \exists y : X.j \Vdash^{\mathrm{K}}_{\mathbb{P}} \varphi[\vec{x}, y];$$

$$j \Vdash^{\mathrm{K}}_{\mathbb{P}} (\forall y : X.\varphi[\vec{x}, y]) \coloneqq \forall k \geq_{\mathbb{P}} j \forall y : X.k(k \Vdash^{\mathrm{K}}_{\mathbb{P}} \varphi[\vec{x}, y]).$$

Proposition

For any $\mathcal{L}_X$-formula $\varphi[\vec{x}]$,

$$\mathcal{E} \models \forall \mathbb{P} \subseteq \mathrm{Lop} \forall j \in \mathrm{Lop} \forall \vec{x} : X.(j(j \Vdash^{\mathrm{K}}_{\mathbb{P}} \varphi[\vec{x}]) \leftrightarrow j \Vdash_{\mathbb{P}} \varphi[\vec{x}]).$$

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- For details on the **effective topos** $\mathcal{E}ff$, refer to van Oosten's textbook and excellent MSc theses by ILLC students.
- An $\mathbf{HA}$-formula $\varphi[\vec{x}]$ is interpreted as a $\mathcal{P}(\mathbb{N})$-valued function $[\![\varphi]\!]$:

$$\varphi\colon N^m \to \Omega \quad \mapsto \quad [\![\varphi]\!]\colon \mathbb{N}^m \to \mathcal{P}(\mathbb{N}).$$

  This interpretation coincides with Kleene realizability in the following sense.

Kleene realizability

Let $n \in \mathbb{N}$.

$$n \mathbf{\ r\ } (\varphi \to \psi) \overset{\text{def}}{\iff} \forall m \in \mathbb{N}.(m \mathbf{\ r\ } \varphi \implies \Phi_n(m) \mathbf{\ r\ } \psi).$$

Proposition

For any $\mathbf{HA}$-sentence φ,

$$[\![\varphi]\!] \neq \emptyset \iff \{\, n \mid n \mathbf{\ r\ } \varphi \,\} \neq \emptyset.$$

Local operators and Lop-frames in $\mathcal{E}ff$

Proposition (Pitts 88)

A function $j\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a **local operator in** $\mathcal{E}ff$ if and only if:

$$[\![\text{is-lop}[j]]\!] := \bigcap_{p,q \subseteq \mathbb{N}} ((p \to j(p)) \wedge (jj(p) \to j(p)) \wedge ((p \to q) \to (j(p) \to j(q)))) \neq \emptyset.$$

Proposition

A function $\mathbb{P}\colon \mathcal{P}(\mathbb{N})^{\mathcal{P}(\mathbb{N})} \to \mathcal{P}(\mathbb{N})$ is a **lop-frame in** $\mathcal{E}ff$ if and only if:

$$\bigcap_{j \in \mathcal{P}(\mathbb{N})^{\mathcal{P}(\mathbb{N})}} (\mathbb{P}(j) \to [\![\text{is-lop}[j]]\!]) \wedge (\text{ “}\mathbb{P}\text{ is relational” }) \neq \emptyset.$$

- For a partial function f, there is a known uniform construction of a local operator j_f.
  (Theoretically, j_f is the least operator making the graph of f dense.)
- The validity of the associated j_f-translation coincides with Kleene realizability relative to f.

relativized Kleene realizability PF : a set of partial functions on $\mathbb{N}$

Let $n \in \mathbb{N}$ and $f \in \mathrm{PF}$.

$$n \,\mathbf{r}^f\, (\varphi \to \psi) \overset{\text{def}}{\iff} \forall m \in \mathbb{N}.(m \,\mathbf{r}^f\, \varphi \implies \Phi_n^f(m) \,\mathbf{r}^f\, \psi).$$

Proposition (essentially, Phoa 89)

For any $\mathbf{HA}$-sentence φ,

$$[\![\varphi^{j_f}]\!] \neq \emptyset \iff \{\, n \mid n \,\mathbf{r}^f\, \varphi \,\} \neq \emptyset.$$

De Jongh-Goodman realizability is a special case

De Jongh-Goodman realizability PF : a set of partial functions on $\mathbb{N}$

Let $n \in \mathbb{N}$, $f \in \mathrm{PF}$, and $T \subseteq \mathrm{PF}$.

$$f \Vdash_T n\ \mathbf{r}\ (\varphi \to \psi) \overset{\text{def}}{\iff} \forall g \in T \forall m \in \mathbb{N}. \\ (f \subseteq g \wedge g \Vdash_T m\ \mathbf{r}\ \varphi \implies g \Vdash_T \Phi_n^g(m)\ \mathbf{r}\ \psi).$$

Theorem (N.)

There exists a uniform construction of a lop-frame $\mathbb{P}_T$ from T.
Assume that T satisfies the following condition:

$$\forall f, g \in T.(f \subseteq g \iff j_f \leq j_g).$$

Then, for any $\mathbf{HA}$-sentence φ and any $f \in T$,

$$[\![j_f \Vdash_{\mathbb{P}_T} \varphi]\!] \neq \emptyset \iff \{ n \mid f \Vdash_T n\ \mathbf{r}\ \varphi \} \neq \emptyset.$$

Application to semi-classical axioms

Consider separation problems of semi-classical axioms. The j-translation in $\mathcal{E}ff$ has a limitation regarding the **double negated variant**:

$$\neg\neg T := \{\, \neg\neg\psi \mid \psi \in T \,\}$$

Proposition

For any local operator j in $\mathcal{E}ff$ and any theory T,

$$\mathcal{E}ff \models (\neg\neg T)^j \implies \mathcal{E}ff \models T^j.$$

Therefore, a theory T and its double negated variant $\neg\neg T$ are never separable by any local operator in $\mathcal{E}ff$.

However, double negated variants appear naturally in intuitionistic proof theory. [Fujiwara & Kurahashi 21] investigated the strength of the Prenex Normal Form Theorem (PNFT) in the hierarchy of semi-classical arithmetic. They focused on the classes $\mathrm{E}_n(\approx$ classical $\Sigma_n)$ and $\mathrm{U}_n(\approx$ classical $\Pi_n)$:

- $\mathbf{HA} + \Pi_n \vee \Pi_n\textbf{-DNE}$ proves the PNFT for U_n.
- $\mathbf{HA} + \Sigma_n\textbf{-DNE} + \neg\neg(\Pi_n \vee \Pi_n\textbf{-DNE})$ proves the PNFT for E_n.
- Furthermore, these axioms are necessary in a precise sense.

Thus, these two theories exhibit distinct properties regarding the PNFT.

Theorem (N.)

$$\mathbf{HA} + \Sigma_n\text{-}\mathbf{DNE} + \neg\neg\Pi_n \vee \Pi_n\text{-}\mathbf{DNE} \nvdash \Pi_n \vee \Pi_n\text{-}\mathbf{DNE} \quad (n \geq 1).$$

Proof (scketch).

Let $\mathbb{P}_n$ be the following lop-frame in $\mathcal{E}ff$:

$$\begin{array}{c} j_{\emptyset^{(n)}} \\ \uparrow \\ j_{\emptyset^{(n-1)}}. \end{array}$$

We write $j := j_{\emptyset^{(n-1)}}$ and $j' := j_{\emptyset^{(n)}}$. Then:

- Since $\mathcal{E}ff \models (\Sigma_n\text{-}\mathbf{DNE})^j$, $\mathcal{E}ff \models j \Vdash_{\mathbb{P}_n} \Sigma_n\text{-}\mathbf{DNE}$.
- Since $\mathcal{E}ff \not\models (\Pi_n \vee \Pi_n\text{-}\mathbf{DNE})^j$, $\mathcal{E}ff \not\models j \Vdash_{\mathbb{P}_n} \Pi_n \vee \Pi_n\text{-}\mathbf{DNE}$.
- But $\mathcal{E}ff \models (\Pi_n \vee \Pi_n\text{-}\mathbf{DNE})^{j'}$, $\mathcal{E}ff \models j \Vdash_{\mathbb{P}_n} \neg\neg\Pi_n \vee \Pi_n\text{-}\mathbf{DNE}$.

By the soundness for $\mathbf{HA}$, we conclude the separation. □

Summary

- We have introduced a translation $j \Vdash_{\mathbb{P}} \varphi$ motivated by the sheaf model of realizability.
- Our translation $j \Vdash_{\mathbb{P}} \varphi$ is sound for $\mathbf{IQC}$ and $\mathbf{HA}$.
- De Jongh-Goodman realizability has been related to $j \Vdash_{\mathbb{P}} \varphi$ in $\mathcal{E}ff$.
- In addition, we have found an application to semi-classical axioms.

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