0-1 laws in graded finite model theory

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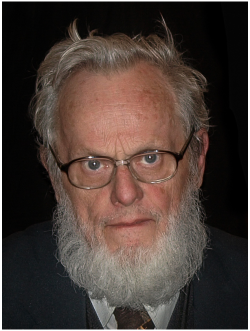
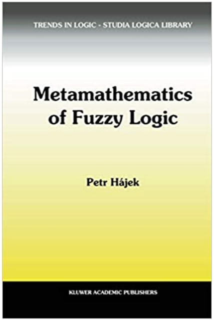
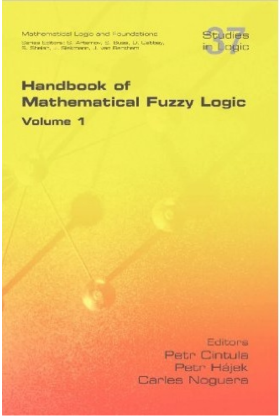
Mathematical Fuzzy Logic

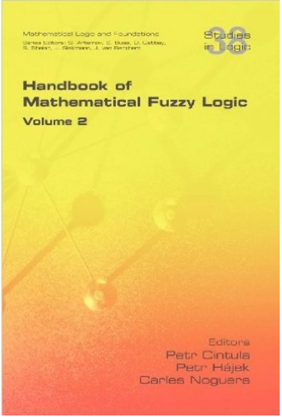
- MFL, as conceived by Petr Hájek (and many others) is a subdiscipline of Mathematical Logic, specializing in the study of certain many-valued logics: those that can be thought as having a linear order of truth-values.
- They're interesting in the study of graded properties, i.e. properties that are a matter of more-or-less such as *red*, *old*, *tall*, or *rich*.
- Examples: Łukasiewicz, Gödel–Dummett, and infinitely-valued Product logics.

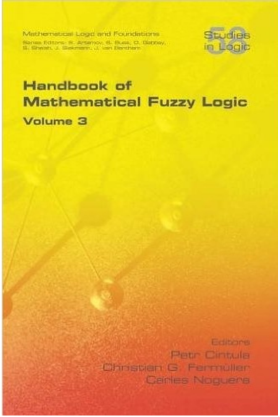




Handbook of Mathematical Fuzzy Logic







Volumes 37, 38 and 58, of Studies in Logic, Mathematical Logic and Foundations, College Publications, 2011 and 2015.

Mathematical Fuzzy Logic: the general setting – 1

- Standard semantics over the unit interval $[0, 1]$.
- Order-based connectives $\vee = \max$ and $\wedge = \min$.
- Truth-constants for total truth ($\overline{1}$) and total falsity ($\overline{0}$).
- Another conjunction & interpreted by a (left-continuous) t-norm: binary commutative, associative, monotone function on $[0, 1]$.
- An implication given by the residuum of the t-norm:

$$a \mathbin{\&} b \leqslant c \text{ if, and only if, } a \leqslant b \rightarrow c.$$

Mathematical Fuzzy Logic: the general setting – 2

Examples of continuous t-norms and their residua:

$$
\begin{aligned}
a \,\&^{[0,1]_{\mathrm{G}}}\, b &= \min\{a, b\}, \\
a \,\&^{[0,1]_{\text{Ł}}}\, b &= \max\{a + b - 1, 0\}, \\
a \,\&^{[0,1]_{\Pi}}\, b &= ab \text{ (standard product of reals)}, \\
a \rightarrow^{[0,1]_{\mathrm{G}}} b &= \begin{cases} 1, & \text{if } a \leqslant b, \\ b, & \text{otherwise}, \end{cases} \\
a \rightarrow^{[0,1]_{\text{Ł}}} b &= \begin{cases} 1, & \text{if } a \leqslant b, \\ 1-a+b, & \text{otherwise}, \end{cases} \\
a \rightarrow^{[0,1]_{\Pi}} b &= \begin{cases} 1, & \text{if } a \leqslant b, \\ b/a, & \text{otherwise}. \end{cases}
\end{aligned}
$$

A t-norm has a residuum iff it is left-continuous.

MTL: logic of left-continuous t-norms.

Algebraic semantics

MTL-algebras are algebraic structures of the form
$\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \&^{\boldsymbol{A}} b \leqslant c \quad \text{iff} \quad b \leqslant a \rightarrow^{\boldsymbol{A}} c, \qquad \text{(residuation)}$$

$$(a \rightarrow^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \rightarrow^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} \qquad \text{(prelinearity)}$$

$\boldsymbol{A}$ is called an MTL-chain if its underlying lattice is linearly ordered.

$\boldsymbol{B}_2$, the two-valued algebra of classical logic, is an extreme example of MTL-chain.

Example: the algebra of Łukasiewicz 3-valued logic

The algebra $\text{Ł}_3 = \langle\{0, \frac{1}{2}, 1\}, \wedge^{\text{Ł}_3}, \vee^{\text{Ł}_3}, \&^{\text{Ł}_3}, \rightarrow^{\text{Ł}_3}, 0, 1\rangle$ such that

- $\wedge^{\text{Ł}_3}(x, y) = \min\{x, y\}$
- $\vee^{\text{Ł}_3}(x, y) = \max\{x, y\}$
- $\&^{\text{Ł}_3}(x, y) = \max\{0, x + y - 1\}$
- $\rightarrow^{\text{Ł}_3}(x, y) = \min\{1, 1 - x + y\}$

Example: the algebra of Gödel 4-valued logic

The algebra $\mathrm{G}_4 = \langle\{0, \frac{1}{3}, \frac{2}{3}, 1\}, \wedge^{\mathrm{G}_4}, \vee^{\mathrm{G}_4}, \&^{\mathrm{G}_4}, \rightarrow^{\mathrm{G}_4}, 0, 1\rangle$ such that

- $\wedge^{\mathrm{G}_4}(x, y) = \&^{\mathrm{G}_4}(x, y) = \min\{x, y\}$
- $\vee^{\mathrm{G}_4}(x, y) = \max\{x, y\}$
- and for $\rightarrow^{\mathrm{G}_4}$:

$$\rightarrow^{\mathrm{G}_4}(x, y) = \begin{cases} \overline{1} & \text{if } x \leqslant y \\ y & \text{otherwise.} \end{cases}$$

Going first-order – 1

- Usual classical syntax with a signature $\tau = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$
- Semantics as in Mostowski, Rasiowa, Hájek tradition $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where:
  - $\boldsymbol{A}$ is an algebra of truth-values (for the propositional language)
  - $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle F_{\mathbf{M}} \rangle_{F \in \mathbf{F}} \rangle$, where
    - M is a non-empty set
    - $F_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary function symbol $F \in \mathbf{F}$.
    - $P_{\mathbf{M}}$ is a function $M^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$
  - An $\mathfrak{M}$-*evaluation* of the object variables is a mapping $\mathrm{v}\colon V \to M$
-
$$
\begin{aligned}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} &= \mathrm{v}(x),\\
\|F(t_1,\dots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}),\\
\|P(t_1,\dots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}),\\
\|\circ(\varphi_1,\dots,\varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}},\dots,\|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}),\\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \inf_{\leqslant_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\},\\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \sup_{\leqslant_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\}.
\end{aligned}
$$

Going first-order – 2

- Notion of safe structure, where truth values of all formulas are defined.
- Notion of model: For each v, $\|\sigma\|^{\mathfrak{M}}_{\mathrm{v}} = \overline{1}^{\boldsymbol{A}}$.

As in classical logic, we have:

- axiomatic Hilbert-style presentation
- completeness theorem

P. Hájek and P. Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

P. Cintula and C.N. A Henkin-Style Proof of Completeness for First-Order Algebraizable Logics, *Journal of Symbolic Logic*, 80(1):341–358, 2015.

Model theory

- Modern logic (and set theory) was developed since the late nineteenth century in connection with important efforts to provide foundations of mathematics [Frege, Cantor, Hilbert, Russell, etc.]
- In the twentieth century mathematical logic evolved into, among others, the branch of model theory: the systematic study of mathematical structures by means of formal languages (mostly, but not only, first-order logic) [Löwenheim, Skolem, Tarski, Fraïssé, Łoś, etc.]
- The second half of the twentieth century has seen a growing interest in the connections between mathematical logic and computer science, which, among many other crucial developments, has justified an increased interest in *finite* mathematical structures.

The goal of this talk – 1

- This talk will link together Mathematical Fuzzy Logic and Finite Model Theory by considering a specific technical problem:

  given a sentence in a first-order fuzzy logic, is there a truth-value that the sentence gets almost surely in finite structures?

- A positive answer to this problem is what we will call a 0-1 law, generalizing a notion from finite model theory based on two-valued logic.

The goal of this talk – 2

- Using different methods, we will generalize the result below to any finitely-valued fuzzy logic:

  Robert Kosik and Christian Fermüller (2009): Almost sure degrees of truth and finite model theory of Łukasiewicz fuzzy logic, *International Journal of the Computer, the Internet and Management* 17 (SP1), 20.1-20.5

- We will also lift the technical restriction in that paper demanding that the number of truth values be a power of 2.

The classical 0-1 law – 1

- In 1950, Rudolf Carnap showed, for finite signatures with only unary relation symbols and a first-order formula φ, that the fraction of structures with domain $\{1, \ldots, n\}$ making φ true always converges to 0 or 1 as n tends to ∞.
- In 1973 (published in the JSL in 1976), Ronald Fagin obtained the result for arbitrary finite relational signatures.

Non-triviality and usefulness of the law

Second-order logic does not satisfy a 0-1 law.

The sentence

$$(\exists X)((\forall x)Xxx \wedge (\forall x, y)(Xxy \rightarrow Xyx) \wedge (\forall x, y, z)((Xxy \wedge Xyz) \rightarrow Xxz)$$

$$\wedge (\forall x)(\exists y)^{=1}(Xxy \wedge y \neq x))$$

is true in exactly the finite structures with an even domain, so the fraction of structures with domain $\{1, \ldots, n\}$ making φ true does not converge to a limit as n tends to ∞.

Hence, we can't axiomatize the class of even structures in first-order logic.

The classical 0-1 law – 2

- 1969: Y.V. Glebskii, D.I. Kogan, M.I. Liogonki and V. Talanov had already proven Fagin's result (though it remained mostly unknown).
- 1980: Talanov proves the 0-1 law for first-order logic with the transitive closure operator.
- 1982: Walter Oberschelp extends Fagin's 0-1 law to parametric classes.
- 1985: Andreas Blass, Yuri Gurevich and Dexter Kozen prove the 0-1 law for first-order logic with the fixpoint operador.
- 1990: Phokion Kolaitis and Moshe Vardi prove the 0-1 law for infinitary first-order logic with bounded variables.
- 1994: Joseph Halpern and Bruce Kapron study the 0-1 law for modal logics and axiomatize the formulas that are true in almost all finite models.

Structures over finite MTL-chains

Fix a finite non-trivial MTL-chain $\boldsymbol{A}$. Safeness is for free.

For each value a of $\boldsymbol{A}$, we will have a truth-constant $\overline{a}$ to denote it. Also, we consider signatures with crisp equality $\approx$.

Structures: $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$.

$\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^s \langle \boldsymbol{A}, \mathbf{N} \rangle$ means that for every τ-sentence σ, $\|\sigma\|^{\langle \boldsymbol{A}, \mathbf{M} \rangle} = \|\sigma\|^{\langle \boldsymbol{A}, \mathbf{N} \rangle}$ (strongly elementarily equivalent structures).

An infinitary language

We call the first-order language described with the above semantics $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$.

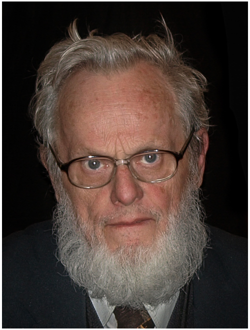
We can close $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ under infinitary lattice disjunctions and conjunctions, e.g. by allowing formulas $\bigwedge_{i\in I}\varphi_i$ and $\bigvee_{i\in I}\varphi_i$ (where I has any cardinality) with the following semantics:

$$\|\textstyle\bigwedge_{i\in I}\varphi_i\|_{\mathbf{M},v}^{\boldsymbol{A}} = \inf\{\|\varphi_i\|_{\mathbf{M},v}^{\boldsymbol{A}} \mid i\in I\};$$
$$\|\textstyle\bigvee_{i\in I}\varphi_i\|_{\mathbf{M},v}^{\boldsymbol{A}} = \sup\{\|\varphi_i\|_{\mathbf{M},v}^{\boldsymbol{A}} \mid i\in I\}.$$

We call the resulting language and semantics $\mathscr{L}_{\infty\omega}^{\boldsymbol{A}}$. If, furthermore, we allow only $k \geqslant 1$ many variables in our formulas, we obtain $\mathscr{L}_{\infty\omega}^{k\boldsymbol{A}}$.

We will give 0-1 laws for $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ and $\mathscr{L}_{\infty\omega}^{k\boldsymbol{A}}$.

Compactness – 1



We have a compactness property for first-order languages with semantics given over a fixed finite MTL-chain (Pilar Dellunde 2014): **every finitely satisfiable set of sentences is satisfiable**.

Compactness is not preserved in general when dealing with infinite MTL algebras: Hájek showed that product predicate logic with the standard semantics on the interval $[0, 1]$ is not compact.

Compactness – 2

Observe that when we restrict ourselves to the study of models with finite domains, compactness breaks apart. It is easy to see that the infinite theory

$$(\forall x_1)(x_1 < x_1 \to \overline{0})$$

$$(\forall x_1, x_2, x_3)(x_1 < x_2 \land x_2 < x_3 \to x_1 < x_3)$$

$$(\exists x_1, \ldots, x_n)\Big(\bigwedge_{1 \leqslant i < j \leqslant n} x_i < x_j\Big) \ \ \text{(for all } n \geqslant 1\text{)}$$

is finitely satisfiable on finite models but not satisfiable.

Partial isomorphisms

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{A}, \mathbf{N} \rangle$ be τ-structures, p be a partial mapping from M to N. We say that p is a partial isomorphism from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{A}, \mathbf{N} \rangle$ if

1. p is injective,
2. for every n-ary functional symbol $F \in \mathbf{F}$ and every $d_1, \ldots, d_n \in M$ such that $d_1, \ldots, d_n, F_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathrm{dom}(p)$,

$$p(F_{\mathbf{M}}(d_1, \ldots d_n)) = F_{\mathbf{N}}(p(d_1), \ldots p(d_n)),$$

3. for every n-ary predicate symbol $P \in \mathbf{P}$ and $d_1, \ldots, d_n \in M$ such that $d_1, \ldots, d_n \in \mathrm{dom}(p)$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) = P_{\mathbf{N}}(p(d_1), \ldots, p(d_n)).$$

Finitely isomorphic structures

Two τ-structures $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{A}, \mathbf{N} \rangle$ are said to be finitely isomorphic, written $\langle \boldsymbol{A}, \mathbf{M} \rangle \cong_f \langle \boldsymbol{A}, \mathbf{N} \rangle$, if there is a sequence $\langle I_n \mid n < \omega \rangle$ with the following properties:

1. Every I_n is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{A}, \mathbf{N} \rangle$.
2. For each $n < \omega$, $I_{n+1} \subseteq I_n$.
3. (Forth-property) For every $p \in I_{n+1}$ and $m \in M$, there is a $p' \in I_n$ such that $p \subseteq p'$ and $m \in \mathrm{dom}(p')$.
4. (Back-property) For every $p \in I_{n+1}$ and $n \in N$, there is a $p' \in I_n$ such that $p \subseteq p'$ and $n \in \mathrm{rg}(p')$.

Proposition 1 (Dellunde, García-Cerdaña, C.N., 2018)

In $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ with τ finite, we have: if $\langle \boldsymbol{A}, \mathbf{M} \rangle \cong_f \langle \boldsymbol{A}, \mathbf{N} \rangle$, then $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^s \langle \boldsymbol{A}, \mathbf{N} \rangle$.

k-potentially isomorphic structures

Given an integer $k \geqslant 1$, two τ-structures $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{A}, \mathbf{N} \rangle$ are said to be k-potentially isomorphic, written $\langle \boldsymbol{A}, \mathbf{M} \rangle \cong^k \langle \boldsymbol{A}, \mathbf{N} \rangle$, if there is a set I of partial isomorphisms with the following properties:

1. I is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{A}, \mathbf{N} \rangle$.
2. I is downward-closed: if $p \in I$ and $p' \subseteq p$, then $p' \in I$.
3. If $p \in I$ and $|\mathrm{dom}(p)| < k$,
    1. (Forth-property) for every $m \in M$, there is a $p' \in I$ such that $p \subseteq p'$ and $m \in \mathrm{dom}(p')$.
    2. (Back-property) for every $n \in N$, there is a $p' \in I$ such that $p \subseteq p'$ and $n \in \mathrm{rg}(p')$.

Proposition 2

Given $k \geqslant 1$ and a finite τ, we have: if $\langle \boldsymbol{A}, \mathbf{M} \rangle \cong^k \langle \boldsymbol{A}, \mathbf{N} \rangle$, then $\|\varphi\|^{\langle \boldsymbol{A}, \mathbf{M} \rangle} = \|\varphi\|^{\langle \boldsymbol{A}, \mathbf{N} \rangle}$ for any sentence of $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$.

Asymptotic probabilities

For any τ-sentence φ, $a \in A$, and $n \geqslant 1$,

$l_n^a(\varphi)$: cardinality of the (finite) set K_τ^a consisting of each model $\mathfrak{M}$ for the signature τ with domain $\{1, 2, \ldots, n\}$ such that $\|\varphi\|^{\mathfrak{M}} = a$.

$l_n(\tau)$: cardinality of the (finite) set containing all model for the signature τ with domain $\{1, 2, \ldots, n\}$.

Now, let

$$\mu_n^a(\varphi) = \frac{l_n^a(\varphi)}{l_n(\tau)}.$$

The asymptotic probability of φ getting value a is defined as follows:

$$\mu^a(\varphi) = \lim_{n \to \infty} \mu_n^a(\varphi).$$

Example 1: crisp predicates

Consider a signature containing a unary relation P. Suppose that $3 \leqslant |A|$. Then,

$$\mu_n^{\bar{1}^{\boldsymbol{A}}}((\forall x)(Px \vee \neg Px)) = \frac{2^n}{|A|^n},$$

so

$$\mu^{\bar{1}^{\boldsymbol{A}}}((\forall x)(Px \vee \neg Px)) = \lim_{n\to\infty} \frac{2^n}{|A|^n} = 0.$$

This means that almost surely no structure makes the predicate P crisp.

Example 2: models with odd domains

Let τ be the empty signature. For any k, let

$$\varphi^{=k} := (\exists x_1, \ldots, x_k)\begin{pmatrix} & \bigwedge_{1\leqslant i<j\leqslant k} x_i \not\approx x_j \\ \wedge & (\forall x_{k+1})(\bigvee_{1\leqslant i\leqslant k} x_{k+1} \approx x_i) \end{pmatrix}.$$

$\mathfrak{M} \models \varphi^{=k}$ iff $|M| = k$. Then, for the infinitary sentence $\bigvee_{k\geqslant 1} \varphi^{=2k+1}$, $\mathfrak{M} \models \bigvee_{k\geqslant 1} \varphi^{=2k+1}$ iff $|M|$ is odd. Then,

$$\mu_n^{\bar{1}^{\boldsymbol{A}}}(\bigvee_{k\geqslant 1} \varphi^{=2k+1}) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

So in this case $\mu^{\bar{1}^{\boldsymbol{A}}}(\bigvee_{k\geqslant 1} \varphi^{=2k+1})$ does not exist.

Example 3: signatures with object constants

Consider a signature containing a unary relation R and an object constant symbol c. Let $a \in A$. Then, for any $n \geqslant 1$,

$$\mu_n^a(Rc) = \frac{1}{|A|}$$

since in any given model $\mathfrak{M}$ with domain $M = \{1, 2, \ldots, n\}$ where we have fixed the interpretation of c, there are $\frac{1}{|A|}$ chances of interpreting R in such a way that $\|Rc\|^{\mathfrak{M}} = a$. Hence,

$$\mu^a(Rc) = \lim_{n \to \infty} \mu_n^a(Rc) = \frac{1}{|A|}.$$

The Rado graph

- Wilhelm Ackermann (1937) introduces first a countably infinite graph as:
  - Symmetrizing membership between hereditarily finite sets
  - Via the BIT predicate between natural numbers: $x \leqslant y$ iff the xth binary bit of y is 1.
- Paul Erdös and Alfréd Rényi (1963) define it as the random graph: for each pair of nodes, put an edge between them with probability 0.5. With probability 1 the resulting graph is isomorphic to the Rado graph.
- Richard Rado (1964) defines it as the universal graph, i.e., it contains all finite graphs as subgraphs. Therefore, it is a Fraïssé limit and saturated.
- Haim Gaifman (1964) axiomatizes the theory a first-order random countable structure and shows that it is ω-categoric (hence, complete).
- Ronald Fagin (1976) uses Gaifman axioms to prove his 0-1 law.

The theory T_τ

Consider a finite relational signature τ. For any $r \geqslant 0$, we let Δ_{r+1} be the (finite) set of all formulas $\varphi(v_1, \ldots, v_r, v_{r+1})$ where φ is an atomic formula $R\overrightarrow{x}$ in the signature τ, v_{r+1} appears in the sequence $\overrightarrow{x}$, and all variables in $\overrightarrow{x}$ are from the list $v_1, \ldots, v_r, v_{r+1}$. Let T_τ be the theory containing, for every A-valued set $\Phi : \Delta_{r+1} \longrightarrow A$, the axiom χ^r_Φ (we will drop the superscript where convenient) defined as:

$$(\forall v_1, \ldots, v_r)(\neg \bigwedge_{1\leqslant i<j\leqslant r} v_i \not\approx v_j \ \vee$$

$$(\exists v_{r+1})\left(\begin{array}{ll} & \bigwedge_{1\leqslant i\leqslant r} v_i \not\approx v_{r+1} \\ \wedge & \bigwedge_{\varphi\in\Delta_{r+1}} (\varphi \leftrightarrow \Phi(\varphi)) \end{array} \right))$$

We call the above an $r+1$ extension axiom of T_τ.

The theory T_τ: key lemma

Lemma 3

Let τ be a finite relational signature. Fix Δ_{r+1} and some A-valued set $\Phi : \Delta_{r+1} \longrightarrow A$. Then, for the extension axiom χ^r_Φ, $\mu^{\overline{1}^{\boldsymbol{A}}}(\chi^r_\Phi) = 1$. In other words, χ^r_Φ takes value $\overline{1}^{\boldsymbol{A}}$ almost surely.

Corollary 4

Let τ be a finite relational signature. For any finite $T'_\tau \subseteq T_\tau$, there is a number k such that for any $n > k$, T'_τ has a model with a universe of objects of size n.

By compactness (in the sense of Dellunde), T_τ has an infinite model.

The case of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$

Proposition 5

Consider a finite signature τ. If $\mathfrak{M}$ and $\mathfrak{N}$ are models of T_τ, then $\mathfrak{M} \cong_f \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are finitely isomorphic.

Corollary 6

For any sentence φ of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ, $T_\tau \vDash \varphi \leftrightarrow \overline{a}$ for some $a \in A$, i.e., any model of T_τ is a model of $\varphi \leftrightarrow \overline{a}$.

The case of $\mathscr{L}^{kA}_{\infty\omega}$

Proposition 7

Consider a finite signature τ. If $\mathfrak{M}$ and $\mathfrak{N}$ are models of χ^r_Φ for all $r \leqslant k$, then $\mathfrak{M} \cong^k \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are k-potentially isomorphic.

Corollary 8

For any sentence φ of $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$ in the signature τ, $T_\tau \vDash \varphi \leftrightarrow \overline{a}$ for some $a \in A$.

The case of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$

Theorem 9 (First 0-1 Law)

If φ is a sentence in the finite relational signature τ, then there is $a \in A$ such that $\mu^{a}(\varphi) = 1$, and for any other truth-value a', $\mu^{a'}(\varphi) = 0$.

The classical version of this result, for $\boldsymbol{B}_2$, simply states that $\mu^1(\varphi) = 1$ or $\mu^1(\varphi) = 0$, or, equivalently, $\mu^0(\varphi) = 1$ or $\mu^0(\varphi) = 0$.

This is immediate from our theorem as 0 or 1 are the only possibilities in that case.

The case of $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$

Theorem 10 (Second 0-1 Law)

If φ is a sentence of $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$ in the finite relational signature τ, then there is $a \in A$ such that $\mu^a(\varphi) = 1$ and for any other truth-value a', $\mu^{a'}(\varphi) = 0$.

$\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$ is, in general, more expressive than its counterpart $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.

Translations of the classical 0-1 law? – 1

Can we import the classical 0-1 law via a translation?

1. There is indeed a translation to two-sorted first-order classical logic:

P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna and C.N., Distinguished Algebraic Semantics For T-Norm Based Fuzzy Logics: Methods and Algebraic Equivalencies, *Annals of Pure and Applied Logic* 160(1):53–81, 2009.

The problem is that this two-sorted language in question contains functions and object constant symbols. So the classical result does not apply to this language.

From the point of view of classical logic, our main results would be claims about very specific two-sorted languages.

Translations of the classical 0-1 law? – 2

2. There is also a translation between many-valued and classical first-order logic (forthcoming in a joint paper with Xavier Caicedo).

Using this translation and a classical 0-1 law for parametric classes (Walter Oberschelp, 1982), we can obtain directly our result.

However, we believe that the direct proofs sketched here are illuminating and help the development of a purely many-valued approach.

Wanna know more?

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10-12 November 2022. Online workshop Finite Model Theory and Many-Valued Logic: Challenges and Interactions.

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