Algebraic and coalgebraic analysis of some many-valued modal logics

Wolfgang Poiger

University of Luxembourg

Joint work with Alexander Kurz and Bruno Teheux

LLAMA Seminar, Amsterdam
March 2024

Many-valued modal logic

- Let **D** be a finite algebra of *truth-degrees* with bounded lattice reduct.

Many-valued modal logic

- Let **D** be a finite algebra of *truth-degrees* with bounded lattice reduct.
- Modal formulas $\varphi \in \text{Form}$ are constructed from propositional variables $p \in \text{Prop}$, the connectives of **D** and modal operators $\Box, \Diamond$.

Many-valued modal logic

- Let **D** be a finite algebra of *truth-degrees* with bounded lattice reduct.
- Modal formulas $\varphi \in \mathrm{Form}$ are constructed from propositional variables $p \in \mathrm{Prop}$, the connectives of **D** and modal operators $\Box, \Diamond$.
- Relational models are $\mathfrak{M} = (W, R, \mathrm{Val})$ with $\mathrm{Val}\colon W \times \mathrm{Prop} \to \mathbf{D}$.

Many-valued modal logic

- Let **D** be a finite algebra of *truth-degrees* with bounded lattice reduct.
- Modal formulas $\varphi \in \mathrm{Form}$ are constructed from propositional variables $p \in \mathrm{Prop}$, the connectives of **D** and modal operators $\Box, \Diamond$.
- Relational models are $\mathfrak{M} = (W, R, \mathrm{Val})$ with $\mathrm{Val}\colon W \times \mathrm{Prop} \to \mathbf{D}$.
- Val is inductively extended to all formulas via the rules

$$\mathrm{Val}(w, \Box\psi) = \bigwedge\{\mathrm{Val}(w', \psi) \mid wRw'\},$$
$$\mathrm{Val}(w, \Diamond\psi) = \bigvee\{\mathrm{Val}(w', \psi) \mid wRw'\}.$$

Many-valued modal logic

- Let **D** be a finite algebra of *truth-degrees* with bounded lattice reduct.
- Modal formulas $\varphi \in \mathrm{Form}$ are constructed from propositional variables $p \in \mathrm{Prop}$, the connectives of **D** and modal operators $\Box, \Diamond$.
- Relational models are $\mathfrak{M} = (W, R, \mathrm{Val})$ with $\mathrm{Val}\colon W \times \mathrm{Prop} \to \mathbf{D}$.
- Val is inductively extended to all formulas via the rules

$$\mathrm{Val}(w, \Box\psi) = \bigwedge\{\mathrm{Val}(w', \psi) \mid wRw'\},$$
$$\mathrm{Val}(w, \Diamond\psi) = \bigvee\{\mathrm{Val}(w', \psi) \mid wRw'\}.$$

- We define $\mathfrak{M}, w \Vdash \varphi$ iff $\mathrm{Val}(w, \varphi) = 1$.

Many-valued modal logic

- Let $\mathbf{D}$ be a finite algebra of *truth-degrees* with bounded lattice reduct.
- Modal formulas $\varphi \in \mathrm{Form}$ are constructed from propositional variables $p \in \mathrm{Prop}$, the connectives of $\mathbf{D}$ and modal operators $\Box, \Diamond$.
- Relational models are $\mathfrak{M} = (W, R, \mathrm{Val})$ with $\mathrm{Val}\colon W \times \mathrm{Prop} \to \mathbf{D}$.
- Val is inductively extended to all formulas via the rules

$$\mathrm{Val}(w, \Box\psi) = \bigwedge\{\mathrm{Val}(w', \psi) \mid wRw'\},$$
$$\mathrm{Val}(w, \Diamond\psi) = \bigvee\{\mathrm{Val}(w', \psi) \mid wRw'\}.$$

- We define $\mathfrak{M}, w \Vdash \varphi$ iff $\mathrm{Val}(w, \varphi) = 1$.
- Recover classical modal logic if $\mathbf{D} = \mathbf{2} \in \mathrm{BA}$.

Examples from many-valued modal logic (1)

Let $\mathbf{D}$ be the $(n+1)$-element finite MV-chain

$$\text{Ł}_n = \langle \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

Hansoul, Teheux 2013 [7] ; Bou, Esteva, Godo, Rodríguez 2011 [1]

Examples from many-valued modal logic (1)

Let $\mathbf{D}$ be the $(n+1)$-element finite MV-chain

$$\mathbf{Ł}_n = \langle \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

For every $d \in Ł_n$, the unary operation $\tau_d \colon Ł_n \to Ł_n$ is term-definable in $\mathbf{Ł}_n$:

$$\tau_d(x) = \begin{cases} 1 & \text{if } x \geq d, \\ 0 & \text{if } x \not\geq d. \end{cases}$$

Hansoul, Teheux 2013 [7] ; Bou, Esteva, Godo, Rodríguez 2011 [1]

Examples from many-valued modal logic (1)

Let $\mathbf{D}$ be the $(n+1)$-element finite MV-chain

$$\mathbf{Ł}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

For every $d \in Ł_n$, the unary operation $\tau_d \colon Ł_n \to Ł_n$ is term-definable in $\mathbf{Ł}_n$:

$$\tau_d(x) = \begin{cases} 1 & \text{if } x \geq d, \\ 0 & \text{if } x \not\geq d. \end{cases}$$

The algebraic counterpart of the corresponding modal logic:

Definition

A *modal MV_n-algebra* is an algebra $(\mathbf{A}, \Box)$ with $\mathbf{A} \in MV_n = \mathbb{HSP}(\mathbf{Ł}_n)$,

- $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$,
- $\Box \tau_d(x) = \tau_d(\Box x)$ for all $d \in Ł_n \backslash \{0\}$.

Hansoul, Teheux 2013 [7] ; Bou, Esteva, Godo, Rodríguez 2011 [1]

Examples from many-valued modal logic (2)

$$\mathbf{H} = \langle H, \wedge, \vee, \rightarrow 0, 1, (T_d \mid d \in H) \rangle,$$

where $\langle H, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a finite Heyting algebra expanded by unary

$$\mathrm{T}_d(x) = \begin{cases} 1 & \text{if } x = d, \\ 0 & \text{if } x \neq d. \end{cases}$$

Note that $\tau_d(x) = \bigvee\{T_c(x) \mid c \geq d\}$ are again term-definable in $\mathbf{H}$.

Maruyama 2009 [13]

Examples from many-valued modal logic (2)

$$\mathbf{H} = \langle H, \wedge, \vee, \to 0, 1, (T_d \mid d \in H)\rangle,$$

where $\langle H, \wedge, \vee, \to, 0, 1\rangle$ is a finite Heyting algebra expanded by unary

$$\mathrm{T}_d(x) = \begin{cases} 1 & \text{if } x = d, \\ 0 & \text{if } x \neq d. \end{cases}$$

Note that $\tau_d(x) = \bigvee\{T_c(x) \mid c \geq d\}$ are again term-definable in $\mathbf{H}$.

The algebraic counterpart of the corresponding modal logic:

Definition

A *modal* $\mathbf{H}$-*algebra* is an algebra $(\mathbf{A}, \Box)$ with $\mathbf{A} \in \mathbb{HSP}(\mathbf{H})$,

- $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$,
- $\Box \tau_d(x) = \tau_d(\Box x)$ for all $d \in H \backslash \{0\}$.

Maruyama 2009 [13]

Examples from many-valued modal logic (3)

Let $\mathbf{D}$ be given by the $(n+1)$-element Łukasiewicz-Moisil chain

$$\mathbf{M}_n = \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_d \mid d \in M_n) \rangle.$$

where $\neg$ is the MV-negation and $\tau_d = \chi_{\{x \geq d\}}$ similar to before.

Diaconescu, Georgescu 2007 [5]

Examples from many-valued modal logic (3)

Let **D** be given by the $(n+1)$-element Łukasiewicz-Moisil chain

$$\mathbf{M}_n = \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_d \mid d \in M_n) \rangle.$$

where $\neg$ is the MV-negation and $\tau_d = \chi_{\{x \geq d\}}$ similar to before.

The algebraic counterpart of the corresponding tense logic:

Definition

A *tense* $Ł M_n$*-algebra* is an algebra $(\mathbf{A}, G, H)$ with $\mathbf{A} \in ŁM_n = \mathbb{HSP}(\mathbf{M}_n)$,

- $G(x \wedge y) = Gx \wedge Gy$ and $G1 = 1$,
- $H(x \wedge y) = Hx \wedge Hy$ and $H1 = 1$,
- $x \leq GPx$ and $x \leq HFx$,
- $G\tau_d(x) = \tau_d(Gx)$ for all $d \in M_n \backslash \{0\}$,
- $H\tau_d(x) = \tau_d(Hx)$ for all $d \in M_n \backslash \{0\}$.

Diaconescu, Georgescu 2007 [5]

Motivating questions

Question: What do these algebras of truth-degrees have in common?

Motivating questions

Question: What do these algebras of truth-degrees have in common?

Question: Is there a general framework to systematically study the relationship between these many-valued modal logics and classical modal logic?

Motivating questions

Question: What do these algebras of truth-degrees have in common?

Answer: They are all *semi-primal*.

Question: Is there a general framework to systematically study the relationship between these many-valued modal logics and classical modal logic?

Motivating questions

Question: What do these algebras of truth-degrees have in common?

Answer: They are all *semi-primal*.

Question: Is there a general framework to systematically study the relationship between these many-valued modal logics and classical modal logic?

Answer: Such a framework is provided by *coalgebraic logic*.

Semi-primal algebras

Definition

An algebra $\mathbf{D}$ is *primal* if every operation $f\colon D^k \to D$ with $k \geq 1$ is term-definable in $\mathbf{D}$.

Foster, Pixley 1964 [6] ; Pixley 1971 [16]

Semi-primal algebras

Definition

An algebra $\mathbf{D}$ is *semi-primal* if every operation $f\colon D^k \to D$ with $k \geq 1$ which preserves subalgebras is term-definable in $\mathbf{D}$.

Foster, Pixley 1964 [6] ; Pixley 1971 [16]

Semi-primal algebras

Definition

An algebra **D** is *semi-primal* if every operation $f\colon D^k \to D$ with $k \geq 1$ which preserves subalgebras is term-definable in **D**.

Theorem

For a finite algebra **D**, t.f.a.e.:

1. **D** is semi-primal.
2. The variety $\mathbb{HSP}(\mathbf{D})$ is arithmetical (*i.e.*, congruence-distributive and -permutable) and all subalgebras of **D** are simple and rigid.

Foster, Pixley 1964 [6] ; Pixley 1971 [16]

Semi-primal algebras

Definition

An algebra **D** is *semi-primal* if every operation $f\colon D^k \to D$ with $k \geq 1$ which preserves subalgebras is term-definable in **D**.

Theorem

For a finite algebra **D**, t.f.a.e.:

1. **D** is semi-primal.
2. The variety $\mathbb{HSP}(\mathbf{D})$ is arithmetical (*i.e.*, congruence-distributive and -permutable) and all subalgebras of **D** are simple and rigid.
3. The ternary discriminator is term-definable in **D** and all subalgebras of **D** are rigid.

Foster, Pixley 1964 [6] ; Pixley 1971 [16]

Semi-primal lattice-expansions

Proposition

For a finite algebra $\mathbf{D}$ with bounded lattice reduct, t.f.a.e.:

1. $\mathbf{D}$ is semi-primal.
2. For every $d \in D$, the unary operation $\tau_d = \chi_{\{x \geq d\}}$ is term-definable and the unary operation $T_0 = \chi_{\{0\}}$ is term-definable.

Semi-primal lattice-expansions

Proposition

For a finite algebra **D** with bounded lattice reduct, t.f.a.e.:

1. **D** is semi-primal.
2. For every $d \in D$, the unary operation $\tau_d = \chi_{\{x \geq d\}}$ is term-definable and the unary operation $T_0 = \chi_{\{0\}}$ is term-definable.

For a finite algebra **D** with bounded *residuated* lattice reduct, t.f.a.e.:

1. **D** is semi-primal.
2. For every $d \in D$, the unary operation $\tau_d = \chi_{\{x \geq d\}}$ is term-definable.

Semi-primal lattice-expansions

Proposition

For a finite algebra $\mathbf{D}$ with bounded lattice reduct, t.f.a.e.:

1. $\mathbf{D}$ is semi-primal.
2. For every $d \in D$, the unary operation $\tau_d = \chi_{\{x \geq d\}}$ is term-definable and the unary operation $T_0 = \chi_{\{0\}}$ is term-definable.

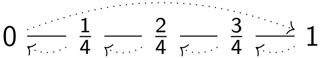
For a finite algebra $\mathbf{D}$ with bounded *residuated* lattice reduct, t.f.a.e.:

1. $\mathbf{D}$ is semi-primal.
2. For every $d \in D$, the unary operation $\tau_d = \chi_{\{x \geq d\}}$ is term-definable.

For the second part, note that we can define $T_0(x) = \tau_e(x \backslash 0)$ where e is the monoid unit of $\mathbf{D}$.

- The Post chains $\mathbf{P}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, ', 0, 1 \rangle$.

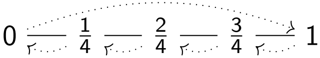
$\mathbf{P}_4$:



Semi-primal chains: Examples

- The Post chains $\mathbf{P}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, ', 0, 1\rangle$.

$\mathbf{P}_4$:

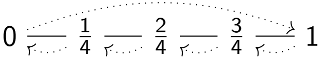


- The finite MV-chains

$$\text{Ł}_n = \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1\rangle.$$

- The Post chains $\mathbf{P}_n = \langle \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \wedge, \vee, ', 0, 1 \rangle$.

$\mathbf{P}_4$:



- The finite MV-chains

$$\text{Ł}_n = \langle \{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

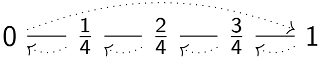
- The finite Łukasiewicz-Moisil chains

$$\mathbf{M}_n = \langle \{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_d \mid d \in M_n) \rangle.$$

Semi-primal chains: Examples

- The Post chains $\mathbf{P}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, ', 0, 1 \rangle$.

$\mathbf{P}_4$:



- The finite MV-chains

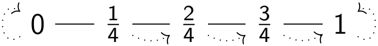
$$\text{Ł}_n = \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

- The finite Łukasiewicz-Moisil chains

$$\mathbf{M}_n = \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_d \mid d \in M_n) \rangle.$$

- The finite Cornish chains $\mathbf{C}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, \neg, f, 0, 1 \rangle$.[1]

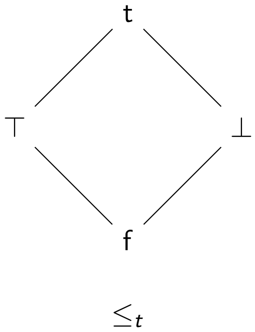
$\mathbf{C}_4$:



[1] Davey, Gair 2017 [4]

Semi-primal lattices: Examples (1)

$$\mathbf{FOUR} = \langle\{t, f, \top, \bot\}, \wedge, \vee, \otimes, \oplus, \neg, \supset, t, f\rangle.$$



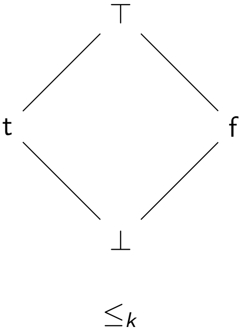
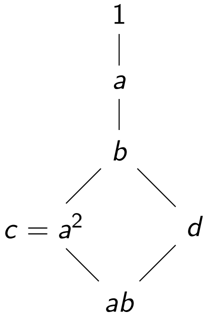


Figure: The truth-order $\leq_t$ and the knowledge-order $\leq_k$.

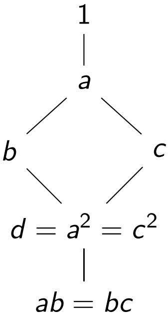
U. Rivieccio's PhD thesis

Semi-primal lattices: Examples (2)

- Residuated lattices, *e.g.*,



$\mathbf{R}^{6,2}_{1,11}$

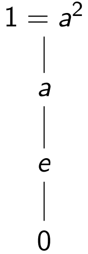


$\mathbf{R}^{6,3}_{1,9}$

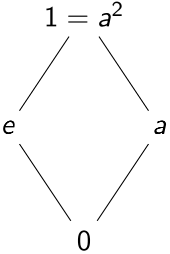
Notation from list of finite residuated lattices of size up to 6 by N. Galatos and P. Jipsen

Semi-primal lattices: Examples (3)

- De Morgan monoids (with unit e) / Relevant algebras (without e)



$\mathbf{C}_4^{01}$



$\mathbf{D}_4^{01}$

Moraschini, Raftery, Wannenburg 2019 [15]

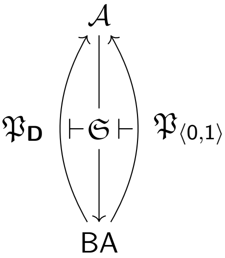
Category-theoretical characterization

Theorem

Let **D** be a bounded lattice-based algebra and $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$. Then **D** is semi-primal if and only if there exists a topological adjunction $\mathfrak{P}_{\mathbf{D}} \vdash \mathfrak{S} \vdash \mathfrak{P}_{\langle 0,1 \rangle}$.

$\mathfrak{P}_{\mathbf{D}}, \mathfrak{P}_{\langle 0,1 \rangle} \colon \mathsf{BA} \to \mathcal{A}$ are *Boolean power functors.*

$\mathfrak{S} \colon \mathcal{A} \to \mathsf{BA}$ is the *Boolean skeleton functor*.



Kurz, P., Teheux 2024 [12]

Algebras and Coalgebras

Let C be a category and let $\mathsf{F}\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor.

$\alpha\colon \mathsf{F}(A) \to A$	$\gamma\colon X \to \mathsf{F}(X)$
F-algebra	F-coalgebra

Algebras and Coalgebras

Let C be a category and let $\mathsf{F}\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor.

$$\alpha\colon \mathsf{F}(A) \to A \qquad\qquad \gamma\colon X \to \mathsf{F}(X)$$

F-algebra F-coalgebra

Morphisms:

$$\begin{array}{ccc} \mathsf{F}(A_1) & \xrightarrow{\alpha_1} & A_1 \\ {\scriptstyle \mathsf{F}h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathsf{F}(A_2) & \xrightarrow[\alpha_2]{} & A_2 \end{array} \qquad\qquad \begin{array}{ccc} X_1 & \xrightarrow{\gamma_1} & \mathsf{F}(X_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{F}f} \\ X_2 & \xrightarrow[\gamma_2]{} & \mathsf{F}(X_2) \end{array}$$

Gives rise to categories Alg(F) and Coalg(F).

Kripke frames as coalgebras

We consider coalgebras for the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$.

Kripke frames as coalgebras

We consider coalgebras for the covariant powerset functor $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$.

A $\mathcal{P}$-coalgebra is a map $\gamma \colon X \to \mathcal{P}(X)$. This can be identified with a relational structure (X, R_γ), where

$$x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1).$$

Kripke frames as coalgebras

We consider coalgebras for the covariant powerset functor $\mathcal{P} \colon \mathrm{Set} \to \mathrm{Set}$.

A $\mathcal{P}$-coalgebra is a map $\gamma \colon X \to \mathcal{P}(X)$. This can be identified with a relational structure (X, R_γ), where

$$x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1).$$

Morphism:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_1} & \mathcal{P}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ Y & \xrightarrow[\gamma_2]{} & \mathcal{P}(Y) \end{array} \qquad \gamma_2(f(x_1)) = f[\gamma_1(x_1)]$$

Kripke frames as coalgebras

We consider coalgebras for the covariant powerset functor $\mathcal{P}\colon \mathrm{Set} \to \mathrm{Set}$.

A $\mathcal{P}$-coalgebra is a map $\gamma\colon X \to \mathcal{P}(X)$. This can be identified with a relational structure (X, R_γ), where

$$x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1).$$

Morphism:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_1} & \mathcal{P}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ Y & \xrightarrow[\gamma_2]{} & \mathcal{P}(Y) \end{array} \qquad \gamma_2(f(x_1)) = f[\gamma_1(x_1)]$$

'$\supseteq$': If $x_1 R_{\gamma_1} x_2$ then $f(x_1) R_{\gamma_2} f(x_2)$.

Kripke frames as coalgebras

We consider coalgebras for the covariant powerset functor $\mathcal{P}\colon \mathrm{Set} \to \mathrm{Set}$.

A $\mathcal{P}$-coalgebra is a map $\gamma\colon X \to \mathcal{P}(X)$. This can be identified with a relational structure (X, R_γ), where

$$x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1).$$

Morphism:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_1} & \mathcal{P}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ Y & \xrightarrow[\gamma_2]{} & \mathcal{P}(Y) \end{array} \qquad \gamma_2(f(x_1)) = f[\gamma_1(x_1)]$$

'$\supseteq$': If $x_1 R_{\gamma_1} x_2$ then $f(x_1) R_{\gamma_2} f(x_2)$.

'$\subseteq$': If $f(x_1) R_{\gamma_2} y$ then $\exists x_2\colon x_1 R_{\gamma_1} x_2$ and $f(x_2) = y$.

Kripke frames as coalgebras

We consider coalgebras for the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$.

A $\mathcal{P}$-coalgebra is a map $\gamma\colon X \to \mathcal{P}(X)$. This can be identified with a relational structure (X, R_γ), where

$$x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1).$$

Morphism:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_1} & \mathcal{P}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ Y & \xrightarrow[\gamma_2]{} & \mathcal{P}(Y) \end{array} \qquad \gamma_2(f(x_1)) = f[\gamma_1(x_1)]$$

'$\supseteq$': If $x_1 R_{\gamma_1} x_2$ then $f(x_1) R_{\gamma_2} f(x_2)$.

'$\subseteq$': If $f(x_1) R_{\gamma_2} y$ then $\exists x_2\colon x_1 R_{\gamma_1} x_2$ and $f(x_2) = y$.

Therefore $\mathsf{Coalg}(\mathcal{P}) \cong \mathsf{Krip}$.

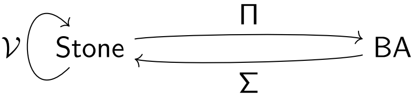
Jónsson-Tarski duality, coalgebraically

$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

Start with Stone duality $\Pi\colon \mathsf{Stone} \to \mathsf{BA}$ (takes clopens) and $\Sigma\colon \mathsf{BA} \to \mathsf{Stone}$ (takes ultrafilters).

Kupke, Kurz, Venema 2003 [10]

Jónsson-Tarski duality, coalgebraically

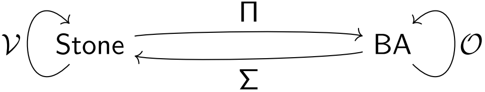


Start with Stone duality $\Pi\colon \mathsf{Stone} \to \mathsf{BA}$ (takes clopens) and $\Sigma\colon \mathsf{BA} \to \mathsf{Stone}$ (takes ultrafilters).

The category of *descriptive general frames* is isomorphic to the category of coalgebras for the Vietoris functor $\mathcal{V}\colon \mathsf{Stone} \to \mathsf{Stone}$.

Kupke, Kurz, Venema 2003 [10]

Jónsson-Tarski duality, coalgebraically



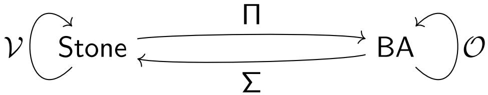
Start with Stone duality $\Pi\colon \mathsf{Stone} \to \mathsf{BA}$ (takes clopens) and $\Sigma\colon \mathsf{BA} \to \mathsf{Stone}$ (takes ultrafilters).

The category of *descriptive general frames* is isomorphic to the category of coalgebras for the Vietoris functor $\mathcal{V}\colon \mathsf{Stone} \to \mathsf{Stone}$.

The category of *modal algebras* is isomorphic to the category of algebras for the functor $\mathcal{O}\colon \mathsf{BA} \to \mathsf{BA}$ which has a *presentation* by a unary operation $\Box$ with equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Kupke, Kurz, Venema 2003 [10]

Jónsson-Tarski duality, coalgebraically



$\delta\colon \mathcal{O}\Pi \Rightarrow \Pi\mathcal{V}$

Start with Stone duality $\Pi\colon \mathsf{Stone} \to \mathsf{BA}$ (takes clopens) and $\Sigma\colon \mathsf{BA} \to \mathsf{Stone}$ (takes ultrafilters).

The category of *descriptive general frames* is isomorphic to the category of coalgebras for the Vietoris functor $\mathcal{V}\colon \mathsf{Stone} \to \mathsf{Stone}$.

The category of *modal algebras* is isomorphic to the category of algebras for the functor $\mathcal{O}\colon \mathsf{BA} \to \mathsf{BA}$ which has a *presentation* by a unary operation $\Box$ with equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Jónsson-Tarski duality: There is a natural isomorphism $\mathcal{O}\Pi \cong \Pi\mathcal{V}$.

Kupke, Kurz, Venema 2003 [10]

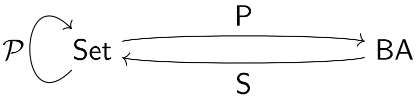
Classical modal logic, coalgebraically

$$\mathsf{Set} \underset{\mathsf{S}}{\overset{\mathsf{P}}{\rightleftarrows}} \mathsf{BA}$$

Begin with dual adjunction $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ (takes powerset) and $\mathsf{S}\colon \mathsf{BA} \to \mathsf{Set}$ (takes ultrafilters).

Kupke, Kurz, Pattinson 2004 [9]

Classical modal logic, coalgebraically

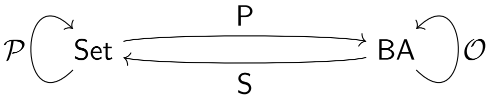


Begin with dual adjunction P: Set $\rightarrow$ BA (takes powerset) and S: BA $\rightarrow$ Set (takes ultrafilters).

The category of *Kripke frames* is isomorphic to the category of coalgebras for the (covariant) powerset functor $\mathcal{P}$: Set $\rightarrow$ Set.

Kupke, Kurz, Pattinson 2004 [9]

Classical modal logic, coalgebraically



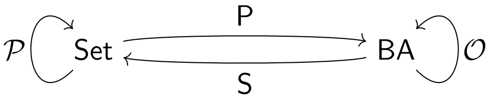
Begin with dual adjunction $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ (takes powerset) and $\mathsf{S}\colon \mathsf{BA} \to \mathsf{Set}$ (takes ultrafilters).

The category of *Kripke frames* is isomorphic to the category of coalgebras for the (covariant) powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$.

The category of *modal algebras* is isomorphic to the category of algebras for the functor $\mathcal{O}\colon \mathsf{BA} \to \mathsf{BA}$ as before.

Kupke, Kurz, Pattinson 2004 [9]

Classical modal logic, coalgebraically



$\delta\colon \mathcal{O}\mathsf{P} \Rightarrow \mathsf{P}\mathcal{P}$

Begin with dual adjunction $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ (takes powerset) and $\mathsf{S}\colon \mathsf{BA} \to \mathsf{Set}$ (takes ultrafilters).

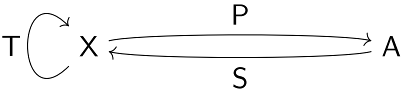
The category of *Kripke frames* is isomorphic to the category of coalgebras for the (covariant) powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$.

The category of *modal algebras* is isomorphic to the category of algebras for the functor $\mathcal{O}\colon \mathsf{BA} \to \mathsf{BA}$ as before.

Sending a Kripke frame to its *complex algebra* can be realized by a *natural transformation* $\mathcal{O}\mathsf{P} \Rightarrow \mathsf{P}\mathcal{P}$.

Kupke, Kurz, Pattinson 2004 [9]

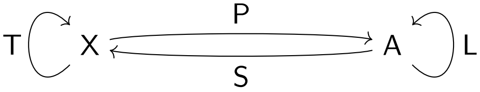
Abstract and concrete coalgebraic logics



Definition

Let X be a concrete category, let A be a variety of algebras, let P and S establish a dual adjunction and let $T\colon X \to X$ be an endofunctor.

Abstract and concrete coalgebraic logics



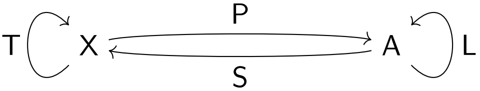
$\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$

Definition

Let X be a concrete category, let A be a variety of algebras, let P and S establish a dual adjunction and let $\mathsf{T}\colon \mathsf{X} \to \mathsf{X}$ be an endofunctor.

1. An *abstract coalgebraic logic* for T is a pair (L, δ) consisting of an endofunctor $\mathsf{L}\colon \mathsf{A} \to \mathsf{A}$ and a natural transformation $\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$.

Abstract and concrete coalgebraic logics



$\delta \colon \mathsf{LP} \Rightarrow \mathsf{PT}$

Definition

Let X be a concrete category, let A be a variety of algebras, let P and S establish a dual adjunction and let $\mathsf{T} \colon \mathsf{X} \to \mathsf{X}$ be an endofunctor.

1. An *abstract coalgebraic logic* for T is a pair (L, δ) consisting of an endofunctor $\mathsf{L} \colon \mathsf{A} \to \mathsf{A}$ and a natural transformation $\delta \colon \mathsf{LP} \Rightarrow \mathsf{PT}$.
2. A *concrete coalgebraic logic* for T is a triple (L, δ, E) consisting of an abstract coalgebraic logic (L, δ) and a presentation E of L by operations and equations.

One-step completeness and expressivity

Definition

An abstract coalgebraic logic (L, δ) for T is *one-step complete* if δ is a monomorphism, *i.e.*, every component of δ is injective.

One-step completeness and expressivity

Definition

An abstract coalgebraic logic (L, δ) for T is *one-step complete* if δ is a monomorphism, *i.e.*, every component of δ is injective.

For example, the abstract coalgebraic logic $(\mathcal{O}, \delta)$ for $\mathcal{P}$ is one-step complete.

One-step completeness and expressivity

Definition

An abstract coalgebraic logic (L, δ) for T is *one-step complete* if δ is a monomorphism, *i.e.*, every component of δ is injective.

For example, the abstract coalgebraic logic $(\mathcal{O}, \delta)$ for $\mathcal{P}$ is one-step complete.

Definition

An abstract coalgebraic logic (L, δ) for T is *expressive* if the adjoint-transpose $\delta^\dagger$ is a component-wise monomorphism.

One-step completeness and expressivity

Definition

An abstract coalgebraic logic (L, δ) for T is *one-step complete* if δ is a monomorphism, *i.e.*, every component of δ is injective.

For example, the abstract coalgebraic logic $(\mathcal{O}, \delta)$ for $\mathcal{P}$ is one-step complete.

Definition

An abstract coalgebraic logic (L, δ) for T is *expressive* if the adjoint-transpose $\delta^\dagger$ is a component-wise monomorphism.

For example, the abstract coalgebraic logic $(\mathcal{O}, \delta)$ for $\mathcal{P}_{\mathrm{fin}}$ is expressive. This is also known as the *Hennessy-Milner property*.

Semi-primal duality

Let **D** be semi-primal bounded lattice-expansion, $\mathcal{A} := \mathbb{HSP}(\mathbf{D}) = \mathbb{ISP}(\mathbf{D})$.

There is a dual equivalence

$$\mathsf{Stone}_{\mathbf{D}} \underset{\Sigma'}{\overset{\Pi'}{\rightleftarrows}} \mathcal{A}$$

Keimel, Werner 1974 [8] ; Clark, Davey 1998 [3]

Semi-primal duality

Let $\mathbf{D}$ be semi-primal bounded lattice-expansion, $\mathcal{A} := \mathbb{HSP}(\mathbf{D}) = \mathbb{ISP}(\mathbf{D})$.

There is a dual equivalence

$$\mathsf{Stone}_{\mathbf{D}} \underset{\Sigma'}{\overset{\Pi'}{\rightleftarrows}} \mathcal{A}$$

Definition

The category $\mathsf{Stone}_{\mathbf{D}}$ has objects (X, v) where $X \in \mathsf{Stone}$ and $v\colon X \to \mathbb{S}(\mathbf{D})$ is continuous w.r.t. the upset topology on $\mathbb{S}(\mathbf{D})$.
A morphism $f\colon (X, v) \to (Y, w)$ is a continuous map $X \to Y$ with $w(f(x)) \leq v(x)$ for all $x \in X$.

Keimel, Werner 1974 [8] ; Clark, Davey 1998 [3]

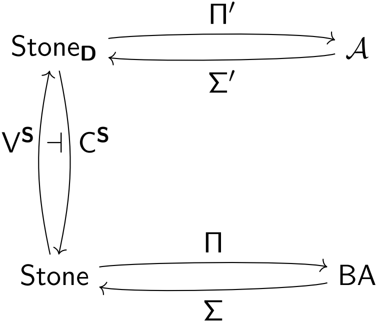
The subalgebra adjunctions

$$\mathsf{Stone}_{\mathbf{D}} \underset{\Sigma'}{\overset{\Pi'}{\rightleftarrows}} \mathcal{A}$$

$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

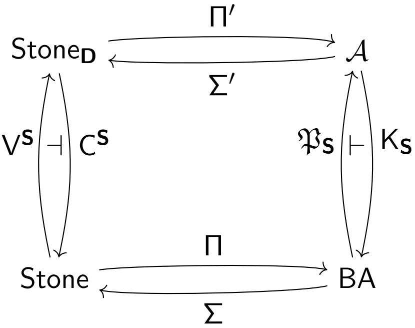
The subalgebra adjunctions

For every $\mathbf{S} \in \mathbb{S}(\mathbf{D})$ there is an adjunction $\mathsf{V}^{\mathbf{S}} \dashv \mathsf{C}^{\mathbf{S}}$.
$\mathsf{V}^{\mathbf{S}}$ sends X to $(X, v^{\mathbf{S}})$ where $v^{\mathbf{S}}$ is constant $\mathbf{S}$.
$\mathsf{C}^{\mathbf{S}}$ sends (X, v) to the closed subspace $\{x \in X \mid v(x) \leq \mathbf{S}\}$.



The subalgebra adjunctions

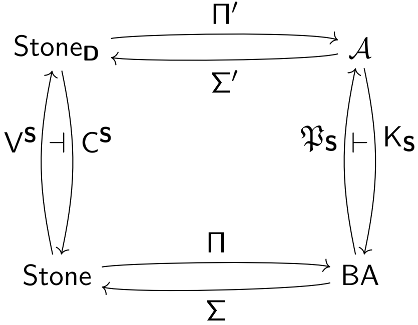
For every $\mathbf{S} \in \mathbb{S}(\mathbf{D})$ there is an adjunction $\mathsf{V}^{\mathbf{S}} \dashv \mathsf{C}^{\mathbf{S}}$.
$\mathsf{V}^{\mathbf{S}}$ sends X to $(X, v^{\mathbf{S}})$ where $v^{\mathbf{S}}$ is constant $\mathbf{S}$.
$\mathsf{C}^{\mathbf{S}}$ sends (X, v) to the closed subspace $\{x \in X \mid v(x) \leq \mathbf{S}\}$.
$\mathfrak{P}_{\mathbf{S}}$, the dual of $\mathsf{V}^{\mathbf{S}}$, takes a *Boolean power*.
$\mathsf{K}_{\mathbf{S}}$, the dual of $\mathsf{C}^{\mathbf{S}}$, takes the *Boolean skeleton* of a quotient.



The subalgebra adjunctions

Any $(X, v) \in \mathsf{Stone}_{\mathbf{D}}$ can be recovered from all $\mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X, v)$ via the *coend*

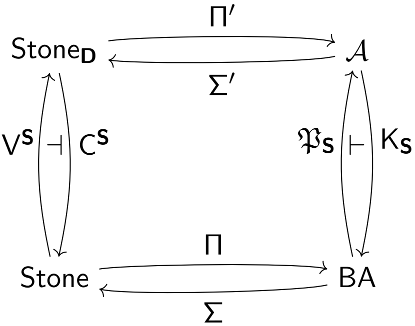
$$(X, v) \cong \int^{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X, v).$$



The subalgebra adjunctions

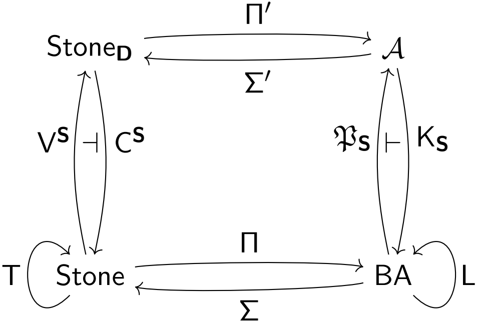
Any $\mathbf{A} \in \mathcal{A}$ can be recovered from all $\mathfrak{P}_{\mathbf{S}}\mathsf{K}_{\mathbf{S}}(\mathbf{A})$ via the *end*

$$\mathbf{A} \cong \int_{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{K}_{\mathbf{S}}(\mathbf{A}).$$



Lifting algebra-coalgebra dualities

Suppose T and L are duals of each other.

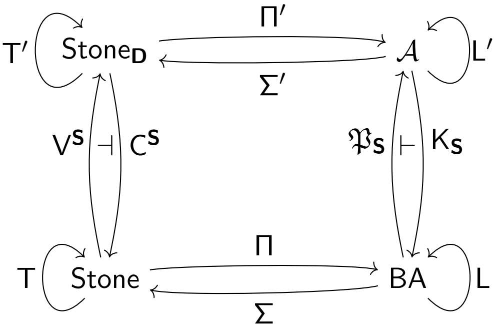


Lifting algebra-coalgebra dualities

Suppose T and L are duals of each other. Define

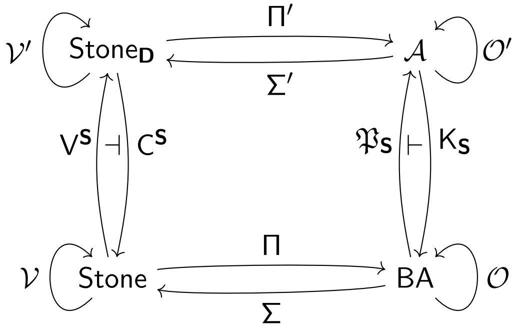
$$\mathsf{T}'(X, v) = \int^{\mathbf{S}} \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X, v) \text{ and } \mathsf{L}'(\mathbf{A}) = \int_{\mathbf{S}} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathsf{K}_{\mathbf{S}}(\mathbf{A}).$$

Then T′ and L′ are duals of each other as well.



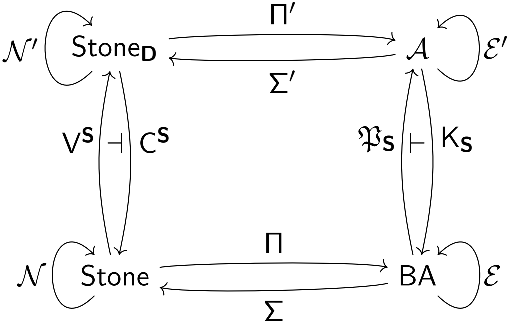
Lifting algebra-coalgebra dualities

For example, this can be used to obtain Maruyama's [14] 'semi-primal version' of Jónsson-Tarski duality from the 'original' Jónsson-Tarski duality.



Lifting algebra-coalgebra dualities

It can also be used to obtain a 'semi-primal version' of Došen duality from the original one as algebra/coalgebra duality described by Bezhanishvilis, de Groot [2]



Forgetting topology

Definition

The category $\mathrm{Stone}_{\mathbf{D}}$ has objects (X, v) where $X \in \mathrm{Stone}$ and $v \colon X \to \mathbb{S}(\mathbf{D})$ is continuous w.r.t. the upset topology on $\mathbb{S}(\mathbf{D})$.
A morphism $f \colon (X, v) \to (Y, w)$ is a continuous map $X \to Y$ with $w(f(x)) \leq v(x)$ for all $x \in X$.

Forgetting topology

Definition

The category $\mathrm{Stone}_{\mathbf{D}}$ has objects (X, v) where $X \in$ Stone and $v \colon X \to \mathbb{S}(\mathbf{D})$ is continuous w.r.t. the upset topology on $\mathbb{S}(\mathbf{D})$.
A morphism $f \colon (X, v) \to (Y, w)$ is a continuous map $X \to Y$ with $w(f(x)) \leq v(x)$ for all $x \in X$.

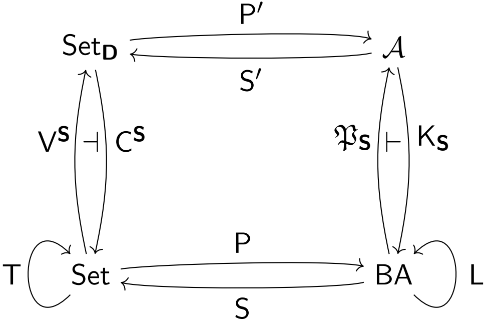
Forgetting topology

Definition

The category $\mathbf{Set_D}$ has objects (X, v) where $X \in \mathbf{Set}$ and $v\colon X \to \mathbb{S}(\mathbf{D})$ ~~is continuous w.r.t. the upset topology on $\mathbb{S}(\mathbf{D})$~~.
A morphism $f\colon (X, v) \to (Y, w)$ is a ~~continuous~~ map $X \to Y$ with $w(f(x)) \leq v(x)$ for all $x \in X$.

Lifting abstract coalgebraic logics

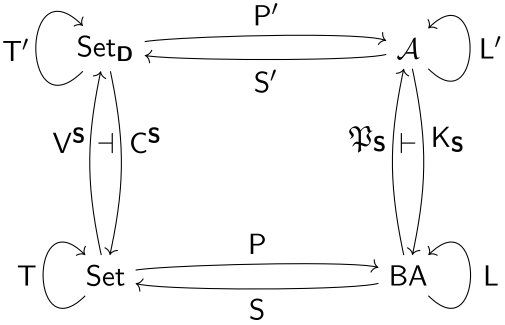
Start with an abstract coalgebraic logic (L, δ) for T.



$\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$

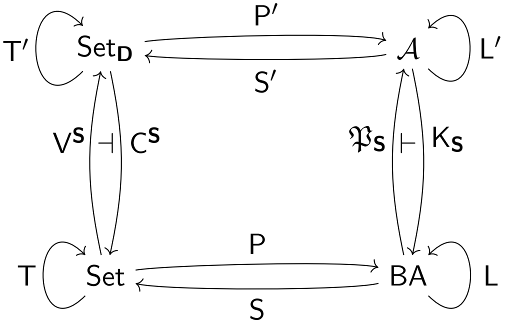
Lifting abstract coalgebraic logics

Start with an abstract coalgebraic logic (L, δ) for T.
Similarly to before, we can lift T and L to T' and L'.



$\delta \colon \mathsf{LP} \Rightarrow \mathsf{PT}$

Start with an abstract coalgebraic logic (L, δ) for T.
Similarly to before, we can lift T and L to T' and L'.
Furthermore, we can define an appropriate δ' from δ.
Thus we obtain a many-valued abstract coalgebraic logic (L', δ') for T'.



$\delta' \colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}'$

$\delta \colon \mathsf{LP} \Rightarrow \mathsf{PT}$

How to obtain δ' from δ

$$\mathsf{L}'\mathsf{P}'(X, v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{LK}_{\mathbf{S}}\mathsf{P}'(X, v) \xrightarrow{\text{limit}} \mathfrak{P}_{\mathbf{S}}\mathsf{LK}_{\mathbf{S}}\mathsf{P}'(X, v)$$

$$\mathsf{P}'\mathsf{T}'(X, v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X, v) \xrightarrow{\text{limit}} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X, v)$$

How to obtain δ' from δ

$$\mathsf{L'P'}(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{LK_{\mathbf{S}}P'}(X,v) \xrightarrow{\text{limit}} \mathfrak{P}_{\mathbf{S}}\mathsf{LK_{\mathbf{S}}P'}(X,v)$$

$$\mathfrak{P}_{\mathbf{S}}\mathsf{LK_{\mathbf{S}}P'}(X,v) \xrightarrow{\cong} \mathfrak{P}_{\mathbf{S}}\mathsf{LPC}^{\mathbf{S}}(X,v)$$

$$\mathfrak{P}_{\mathbf{S}}\mathsf{PTC}^{\mathbf{S}}(X,v) \xrightarrow{\cong} \mathsf{P'V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v)$$

$$\mathsf{P'T'}(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P'V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v) \xrightarrow{\text{limit}} \mathsf{P'V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v)$$

How to obtain δ' from δ

$$\begin{array}{ccc}
\mathsf{L}'\mathsf{P}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathsf{S}}\mathsf{L}\mathsf{K}_{\mathsf{S}}\mathsf{P}'(X,v) & \xrightarrow{\text{limit}} & \mathfrak{P}_{\mathsf{S}}\mathsf{L}\mathsf{K}_{\mathsf{S}}\mathsf{P}'(X,v) \\
 & & \big\downarrow \cong \\
 & & \mathfrak{P}_{\mathsf{S}}\mathsf{LPC}^{\mathsf{S}}(X,v) \\
 & & \big\downarrow \mathfrak{P}_{\mathsf{S}}\delta\mathsf{C}^{\mathsf{S}} \\
 & & \mathfrak{P}_{\mathsf{S}}\mathsf{PTC}^{\mathsf{S}}(X,v) \\
 & & \big\downarrow \cong \\
\mathsf{P}'\mathsf{T}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P}'\mathsf{V}^{\mathsf{S}}\mathsf{TC}^{\mathsf{S}}(X,v) & \xrightarrow{\text{limit}} & \mathsf{P}'\mathsf{V}^{\mathsf{S}}\mathsf{TC}^{\mathsf{S}}(X,v)
\end{array}$$

How to obtain δ' from δ

$$
\begin{array}{ccc}
\mathsf{L}'\mathsf{P}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathsf{K}_{\mathbf{S}}\mathsf{P}'(X,v) & \xrightarrow{\text{limit}} & \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathsf{K}_{\mathbf{S}}\mathsf{P}'(X,v) \\
 & \searrow^{\text{wedge}} & \downarrow \cong \\
 & & \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathsf{P}\mathsf{C}^{\mathbf{S}}(X,v) \\
 & & \downarrow \mathfrak{P}_{\mathbf{S}}\delta\mathsf{C}^{\mathbf{S}} \\
 & & \mathfrak{P}_{\mathbf{S}}\mathsf{P}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v) \\
 & & \downarrow \cong \\
\mathsf{P}'\mathsf{T}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v) & \xrightarrow{\text{limit}} & \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v)
\end{array}
$$

How to obtain δ' from δ

$$
\begin{array}{ccc}
\mathsf{L'P'}(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{LK}_{\mathbf{S}}\mathsf{P'}(X,v) & \xrightarrow{\text{limit}} & \mathfrak{P}_{\mathbf{S}}\mathsf{LK}_{\mathbf{S}}\mathsf{P'}(X,v) \\
 & & \downarrow \cong \\
 & & \mathfrak{P}_{\mathbf{S}}\mathsf{LPC}^{\mathbf{S}}(X,v) \\
\exists!\delta'_{(X,v)} \Big\downarrow & \searrow \text{wedge} & \downarrow \mathfrak{P}_{\mathbf{S}}\delta\mathsf{C}^{\mathbf{S}} \\
 & & \mathfrak{P}_{\mathbf{S}}\mathsf{PTC}^{\mathbf{S}}(X,v) \\
 & & \downarrow \cong \\
\mathsf{P'T'}(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P'V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v) & \xrightarrow{\text{limit}} & \mathsf{P'V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v)
\end{array}
$$

One-step completeness and expressivity

Theorem

Let (L', δ') be the lifting of (L, δ) as on the previous slides.

1. If (L, δ) is one-step complete, then (L', δ') is one-step complete.

One-step completeness and expressivity

Theorem

Let (L', δ') be the lifting of (L, δ) as on the previous slides.

1. If (L, δ) is one-step complete, then (L', δ') is one-step complete.
2. If (L, δ) is expressive, then (L', δ') is expressive.

One-step completeness and expressivity

Theorem

Let (L', δ') be the lifting of (L, δ) as on the previous slides.

1. If (L, δ) is one-step complete, then (L', δ') is one-step complete.
2. If (L, δ) is expressive, then (L', δ') is expressive.
3. If L has a presentation by operations and equations, then L' has one.

One-step completeness and expressivity

Theorem

Let (L', δ') be the lifting of (L, δ) as on the previous slides.

1. If (L, δ) is one-step complete, then (L', δ') is one-step complete.
2. If (L, δ) is expressive, then (L', δ') is expressive.
3. If L has a presentation by operations and equations, then L' has one.

Corollary

If (L, δ) is one-step complete/expressive, then so is $(\mathsf{L}', \delta^{\top})$.

$$\mathsf{T} \circlearrowright \mathsf{Set} \underset{\mathsf{C}^{\mathbf{D}}}{\overset{\mathsf{V}^{\mathbf{D}}}{\rightleftarrows}} \mathsf{Set}_{\mathbf{D}} \underset{\mathsf{S}'}{\overset{\mathsf{P}'}{\rightleftarrows}} \mathcal{A} \circlearrowleft \mathsf{L}' \qquad \delta^{\top} = \delta' \mathsf{V}^{\top}$$

One-step completeness and expressivity

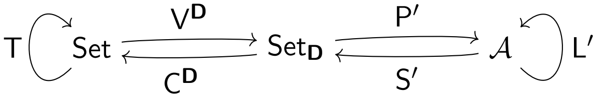
Theorem

Let (L', δ') be the lifting of (L, δ) as on the previous slides.

1. If (L, δ) is one-step complete, then (L', δ') is one-step complete.
2. If (L, δ) is expressive, then (L', δ') is expressive.
3. If L has a presentation by operations and equations, then L' has one.

Corollary

If (L, δ) is one-step complete/expressive, then so is $(\mathsf{L}', \delta^\top)$.



$$\delta^\top = \delta' \mathsf{V}^\top$$

Lifting concrete coalgebraic logics (1)

$$\tau_d(x) = \begin{cases} 1 & \text{if } x \geq d \\ 0 & \text{if } x \not\geq d. \end{cases}$$

Theorem

Let $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ have a presentation by one unary operation $\Box$ and equations which all hold in $\mathbf{D}$ if $\Box$ is replaced by any τ_d, including the equation $\Box(x \wedge y) = \Box x \wedge \Box y$.
Then L' has a presentation by one unary operation $\Box'$ and the following equations.

- $\Box'$ satisfies all equations which the original $\Box$ satisfies,
- $\Box' \tau_d(x) = \tau_d(\Box' x)$ for all $d \in D \setminus \{0\}$.

Lifting concrete coalgebraic logics (2)

$$\tau_d^\partial(x) = \begin{cases} 0 & \text{if } x \leq d \\ 1 & \text{if } x \not\leq d. \end{cases}$$

Theorem

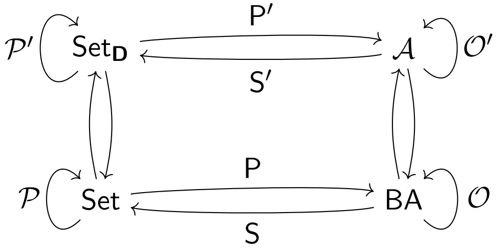
Let $\mathrm{L}\colon \mathrm{BA} \to \mathrm{BA}$ have a presentation by one unary operation $\Diamond$ and equations which all hold in $\mathbf{D}$ if $\Diamond$ is replaced by any τ_d^∂, including the equation $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.

Then L' has a presentation by one unary operation $\Diamond'$ and the following equations.

- $\Diamond'$ satisfies all equations which the original $\Diamond$ satisfies,
- $\Diamond' \tau_d^\partial(x) = \tau_d^\partial(\Diamond' x)$ for all $d \in D \backslash \{1\}$.

Many-valued modal logic as lifting of classical modal logic

The functor $\mathcal{O}$ has a presentation by $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

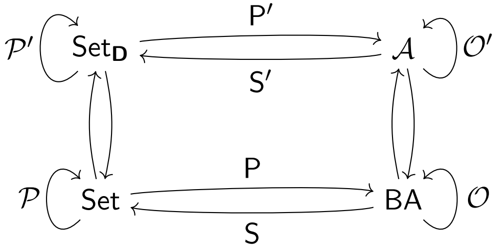


Many-valued modal logic as lifting of classical modal logic

The functor $\mathcal{O}$ has a presentation by $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Therefore, the functor $\mathcal{O}'$ has a presentation by

$\Box'(x \wedge y) = \Box' x \wedge \Box' y$, $\Box' 1 = 1$ and $\Box' \tau_d(x) = \tau_d(\Box' x)$ for all $d \in D \backslash \{0\}$.



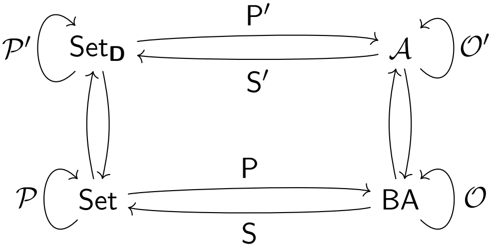
Many-valued modal logic as lifting of classical modal logic

The functor $\mathcal{O}$ has a presentation by $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Therefore, the functor $\mathcal{O}'$ has a presentation by

$\Box'(x \wedge y) = \Box' x \wedge \Box' y$, $\Box' 1 = 1$ and $\Box' \tau_d(x) = \tau_d(\Box' x)$ for all $d \in D \setminus \{0\}$.

$(\mathcal{O}, \delta)$ is (one-step) complete $\Rightarrow$ $(\mathcal{O}', \delta')$ is (one-step) complete.



Many-valued modal logic as lifting of classical modal logic

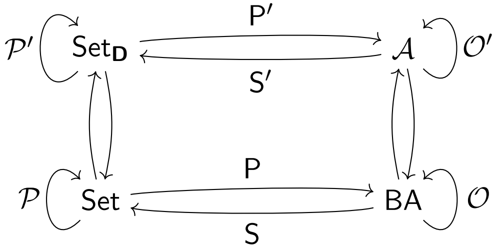
The functor $\mathcal{O}$ has a presentation by $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Therefore, the functor $\mathcal{O}'$ has a presentation by

$\Box'(x \wedge y) = \Box' x \wedge \Box' y$, $\Box' 1 = 1$ and $\Box' \tau_d(x) = \tau_d(\Box' x)$ for all $d \in D \setminus \{0\}$.

$(\mathcal{O}, \delta)$ is (one-step) complete $\Rightarrow$ $(\mathcal{O}', \delta')$ is (one-step) complete.

Replacing $\mathcal{P}$ by $\mathcal{P}_{\mathrm{fin}}$: $(\mathcal{O}, \delta)$ is expressive $\Rightarrow$ $(\mathcal{O}', \delta')$ is expressive.



Lifted semantics

Definition

A $\mathsf{Set}_{\mathbf{D}}$-*(Kripke-)frame* is a structure (W, v, R) with $v\colon X \to \mathbb{S}(\mathbf{D})$ and binary relation $R \subseteq W^2$ satisfying

$$wRw' \Rightarrow v(w') \subseteq v(w)$$

for all $w, w' \in W$.

Lifted semantics

Definition

A $\mathsf{Set}_{\mathbf{D}}$-*(Kripke-)frame* is a structure (W, v, R) with $v\colon X \to \mathbb{S}(\mathbf{D})$ and binary relation $R \subseteq W^2$ satisfying

$$wRw' \Rightarrow v(w') \subseteq v(w)$$

for all $w, w' \in W$.
A $\mathsf{Set}_{\mathbf{D}}$-*model* adds a valuation $\mathsf{Val}\colon W \times \mathsf{Prop} \to \mathbf{D}$ which satisfies

$$\mathsf{Val}(w, p) \in v(w)$$

for all $w \in W$.

Lifted semantics

Definition

A $\mathsf{Set}_{\mathbf{D}}$-*(Kripke-)frame* is a structure (W, v, R) with $v\colon X \to \mathbb{S}(\mathbf{D})$ and binary relation $R \subseteq W^2$ satisfying

$$wRw' \Rightarrow v(w') \subseteq v(w)$$

for all $w, w' \in W$.

A $\mathsf{Set}_{\mathbf{D}}$-*model* adds a valuation $\mathsf{Val}\colon W \times \mathsf{Prop} \to \mathbf{D}$ which satisfies

$$\mathsf{Val}(w, p) \in v(w)$$

for all $w \in W$.

For example, if $\mathbf{D} = \text{Ł}_2$ is the three-element MV-chain, the formula

$$\Diamond(p \vee \neg p).$$

is satisfied in a $\mathsf{Set}_{\mathbf{D}}$-frame if and only if $\forall w \exists w'\colon wRw' \wedge v(w') = \mathbf{2}$, while it is not satisfied in any frame.

Alternative axiomatizations: Some case studies (1)

If $\mathbf{D} = \text{Ł}_n$ is a finite MV-chain, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box(x \oplus x) = \Box x \oplus \Box x$,

(B4) $\Box(x \odot x) = \Box x \odot \Box x$.

Alternative axiomatizations: Some case studies (1)

If $\mathbf{D} = \text{Ł}_n$ is a finite MV-chain, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box(x \oplus x) = \Box x \oplus \Box x$,

(B4) $\Box(x \odot x) = \Box x \odot \Box x$.

If $\mathbf{D}$ is a finite bounded residuated lattice with τ_e (monoid unit e) and truth-constants, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\tau_e(\Box x) = \Box \tau_e(x)$,

(B4) $\Box(r \backslash x) = r \backslash \Box x$ for all $r \neq 0$.

Alternative axiomatizations: Some case studies (1)

If $\mathbf{D} = \text{Ł}_n$ is a finite MV-chain, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box(x \oplus x) = \Box x \oplus \Box x$,

(B4) $\Box(x \odot x) = \Box x \odot \Box x$.

If $\mathbf{D}$ is a finite bounded residuated lattice with τ_e (monoid unit e) and truth-constants, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\tau_e(\Box x) = \Box \tau_e(x)$,

(B4) $\Box(r \backslash x) = r \backslash \Box x$ for all $r \neq 0$.

In particular, if $\mathbf{D}$ is a finite FL_{ew}-algebra with truth-constants where only $0, 1$ are idempotent, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box(x \odot x) = \Box x \odot \Box x$,

(B4) $\Box(r \rightarrow x) = r \rightarrow \Box x$ all $r \neq 0$.

Alternative axiomatizations: Some case studies (2)

If **D** is a finite bi-Heyting algebra with truth-constants and with a unique atom and coatom, then $\mathcal{O}'$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box\big(\neg(1 \leftarrow x)\big) = \neg(1 \leftarrow \Box x)$,

(B4) $\Box(b \rightarrow x) = b \rightarrow \Box x$ all $b \neq 0$,

(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$,

(D1) $\Diamond 0 = 0$,

(D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

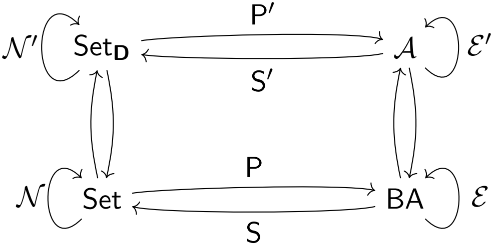
(D3) $\Diamond\big(1 \leftarrow (\neg x)\big) = 1 \leftarrow (\neg \Diamond x)$,

(D4) $\Diamond(x \leftarrow b) = \Diamond x \leftarrow b$ all $b \neq 1$,

(P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

Many-valued modal logic for crisp neighborhoods

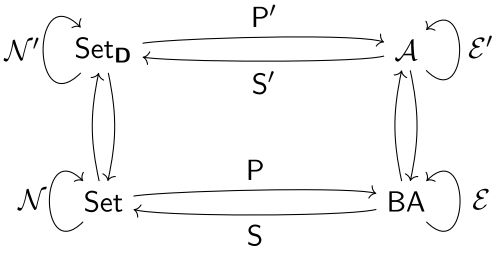
The neighborhood functor $\mathcal{N}$ is the contravariant powerset functor composed with itself. The functor $\mathcal{E}$ has a presentation by one unary operation $\Box$ and *no* equations.



Many-valued modal logic for crisp neighborhoods

The neighborhood functor $\mathcal{N}$ is the contravariant powerset functor composed with itself. The functor $\mathcal{E}$ has a presentation by one unary operation $\Box$ and *no* equations.

$(\mathcal{E}, \delta)$ is (one-step) complete $\Rightarrow$ $(\mathcal{E}', \delta')$ is (one-step) complete.

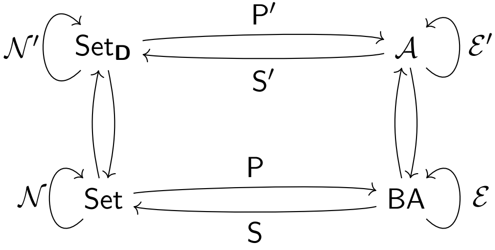


Many-valued modal logic for crisp neighborhoods

The neighborhood functor $\mathcal{N}$ is the contravariant powerset functor composed with itself. The functor $\mathcal{E}$ has a presentation by one unary operation $\Box$ and *no* equations.

$(\mathcal{E}, \delta)$ is (one-step) complete $\Rightarrow$ $(\mathcal{E}', \delta')$ is (one-step) complete.

Replacing $\mathcal{N}$ by $\mathcal{N}_{\mathrm{fin}}$: $(\mathcal{E}, \delta)$ is expressive $\Rightarrow$ $(\mathcal{E}', \delta')$ is expressive.



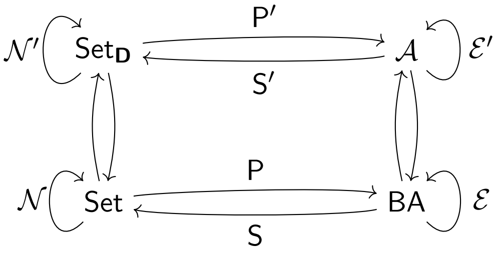
Many-valued modal logic for crisp neighborhoods

The neighborhood functor $\mathcal{N}$ is the contravariant powerset functor composed with itself. The functor $\mathcal{E}$ has a presentation by one unary operation $\Box$ and *no* equations.

$(\mathcal{E}, \delta)$ is (one-step) complete $\Rightarrow$ $(\mathcal{E}', \delta')$ is (one-step) complete.

Replacing $\mathcal{N}$ by $\mathcal{N}_{\mathrm{fin}}$: $(\mathcal{E}, \delta)$ is expressive $\Rightarrow$ $(\mathcal{E}', \delta')$ is expressive.

We don't know a concrete presentation for $\mathcal{E}'$, unless **D** is primal [11]



Conclusion

- *Semi-primal lattice expansions* are ‘well-behaved’ algebras of truth-degrees **D** for many-valued coalgebraic logics.

Conclusion

- *Semi-primal lattice expansions* are 'well-behaved' algebras of truth-degrees **D** for many-valued coalgebraic logics.
- 'Semi-primal versions' of algebra/coalgebra dualities like Jónsson-Tarski or Došen duality can be obtained from their classical counterparts.

Conclusion

- *Semi-primal lattice expansions* are 'well-behaved' algebras of truth-degrees **D** for many-valued coalgebraic logics.
- 'Semi-primal versions' of algebra/coalgebra dualities like Jónsson-Tarski or Došen duality can be obtained from their classical counterparts.
- Similarly, 'semi-primal versions' of coalgebraic logics can be obtained from their classical counterparts.

Conclusion

- *Semi-primal lattice expansions* are 'well-behaved' algebras of truth-degrees **D** for many-valued coalgebraic logics.
- 'Semi-primal versions' of algebra/coalgebra dualities like Jónsson-Tarski or Došen duality can be obtained from their classical counterparts.
- Similarly, 'semi-primal versions' of coalgebraic logics can be obtained from their classical counterparts.
- One-step completeness, expressivity and finite axiomatizability are preserved under this process.

Conclusion

- *Semi-primal lattice expansions* are 'well-behaved' algebras of truth-degrees **D** for many-valued coalgebraic logics.
- 'Semi-primal versions' of algebra/coalgebra dualities like Jónsson-Tarski or Došen duality can be obtained from their classical counterparts.
- Similarly, 'semi-primal versions' of coalgebraic logics can be obtained from their classical counterparts.
- One-step completeness, expressivity and finite axiomatizability are preserved under this process.
- Sometimes, one may obtain axiomatizations of the lifted many-valued logic directly from an axiomatization of the original classical one. In particular, this works for classical modal logic.

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal = Finite discriminator algebras

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal $=$ Finite discriminator algebras
- Lattice-(semi-)primal algebras

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal = Finite discriminator algebras
- Lattice-(semi-)primal algebras
- Infinite algebras, *e.g.*, standard MV-chain $[0,1]$

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal = Finite discriminator algebras
- Lattice-(semi-)primal algebras
- Infinite algebras, *e.g.*, standard MV-chain $[0, 1]$

Investigate broader classes of logics, *e.g.*

- Many-valued modal logic with many-valued accessibility relation

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal = Finite discriminator algebras
- Lattice-(semi-)primal algebras
- Infinite algebras, *e.g.*, standard MV-chain $[0, 1]$

Investigate broader classes of logics, *e.g.*

- Many-valued modal logic with many-valued accessibility relation
- Positive Modal Logic

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal $=$ Finite discriminator algebras
- Lattice-(semi-)primal algebras
- Infinite algebras, *e.g.*, standard MV-chain $[0, 1]$

Investigate broader classes of logics, *e.g.*

- Many-valued modal logic with many-valued accessibility relation
- Positive Modal Logic
- Probabilistic Logic

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal = Finite discriminator algebras
- Lattice-(semi-)primal algebras
- Infinite algebras, *e.g.*, standard MV-chain $[0, 1]$

Investigate broader classes of logics, *e.g.*

- Many-valued modal logic with many-valued accessibility relation
- Positive Modal Logic
- Probabilistic Logic
- Dynamic Logic

The end

Thanks for your attention!

Preprint:
Kurz, A., Poiger, W., and Teheux, B.: *Many-valued coalgebraic logic over semi-primal varieties*
https://arxiv.org/abs/2308.14581

[1] Bou, F., Esteva, F., Godo, L., Rodríguez, R. O.: On the minimum many-valued modal logic over a finite residuated lattice.
Journal of Logic and Computation **21**, 739–790 (2011)
doi:10.1093/logcom/exp062

[2] Bezhanishvili, G., Bezhanishvili, N., de Groot, J.: A Coalgebraic Approach to Dualities for Neighborhood Frames.
Logical Methods in Computer Science **18**, 4:1–4:39 (2022)
doi:10.46298/lmcs-18(3:4)2022

[3] Clark, D. M., Davey, B. A.: Natural Dualities for the Working Algebraist. Cambridge studies in advanced mathematics, vol. 57.
Cambridge University Press (1998)

[4] Davey, B. A., Gair, A.: Restricted Prietley dualities and discriminator varieties.
Studia Logica **105**, 843–872 (2017) ; doi:10.1007/s11225-017-9713-4

[5] Diaconescu, D., Georgescu, G.: Tense operators on MV-algebras and Łukasiewicz-Moisil algebras.
Fundamenta Informaticae **81**, 379–408 (2007)

[6] Foster, A. L., Pixley, A. F.: Semi-categorical algebras. I. Semi-primal algebras.
Mathematische Zeitschrift **83**, 147–169 (1964)
doi:10.1007/BF01111252

References II

[7] Hansoul, G., Teheux, B.: Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics.
Studia Logica **101**, 505–545 (2013)
doi:10.1007/s11225-012-9396-9

[8] Keimel, K., Werner, H.: Stone duality for varieties generated by quasi-primal algebras.
Memoirs of the American Mathematical Society **148**, 59–85 (1974)

[9] Kupke, C., Kurz, A., Pattinson, D.: Algebraic semantics for coalgebraic logics.
Electronic Notes in Theoretical Computer Science **106**, 219–241 (2004)
doi:10.1016/j.entcs.2004.02.037

[10] Kupke, C., Kurz, A., Venema, Y.: Stone coalgebras.
Theoretical Computer Science **327**, 109–134 (2003)
doi:10.1016/S1571-0661(04)80638-8

[11] Kurz, A., Poiger, W.: Many-valued coalgebraic logic: From Boolean algebras to primal varieties. In: 10th Conference on Algebra and Coalgebra in Computer Science (CALCO 2023). Leibniz International Proceedings in Informatics (LIPIcs), **270**, 17:1-17:17, Dagstuhl (2023)
doi:10.4230/LIPIcs.CALCO.2023.17

References III

[12] Kurz, A., Poiger, W., Teheux, B.: New perspectives on semi-primal varieties. Journal of Pure and Applied Algebra**228**, paper no. 107525 (2024)
doi:10.1016/j.jpaa.2023.107525

[13] Maruyama, Y.: Algebraic study of lattice-valued logic and lattice-valued modal logic. In: Ramanujam, R., Sarukkai, S. (eds) Logic and Its Applications. ICLA, 170–184. Springer, Berlin Heidelberg (2009)
doi:10.1007/978-3-540-92701-3_12

[14] Maruyama, Y.: Natural duality, modality, and coalgebra.
Journal of Pure and Applied Algebra **216**, 565–580 (2012)
doi:10.1016/j.jpaa.2011.07.002

[15] Moraschini, T., Raftery, J. G., Wanneburg, J. J.: Varieties of De Morgan monoids: Minimality and irreducible algebras.
Journal of Pure and Applied Algebra **227**, 2780-2803 (2019)
doi:10.1016/j.jpaa.2018.09.015

[16] Pixley, A. F.: The ternary discriminator in universal algebra.
Mathematische Annalen **191**, 167–180 (1971)
doi:doi:10.1016/0012-365X(79)90096-7

[17] Poiger, W.: Natural dualities for varieties generated by finite positive MV-chains. (2023)
Preprint available at https://aps.arxiv.org/abs/2309.16998