Substructural Logics, Fast and Slow

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Recent complexity results and methods for substructural logics

Part I: Upper bounding using well-quasi-orderings

Part II: Axiomatising a tighter upper bound argument for fragments

Joint work

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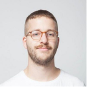

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Part I: Upper bounding using well-quasi-orderings

Substructural logics: remove some structural properties...

...contraction, weakening, exchange, ...

Consider a sequent calculus for intuitionistic logic with explicit rules for above properties

$$\frac{X, U, V, Y \Rightarrow C}{X, V, U, Y \Rightarrow C}\,\mathrm{e} \qquad \frac{X \Rightarrow C}{X, U \Rightarrow C}\,\mathrm{w} \qquad \frac{X, U, U \Rightarrow C}{X, U \Rightarrow C}\,\mathrm{c}$$

The intuitionistic calculus is essentially what is known as FLewc

Now delete any subset of $\{e, w, c\}$ to get FLew, FLec, FLc...

More logics through axioms e.g., $MTL = \text{FLew} + (p \to q) \lor (q \to p)$

Decidability and upper bounding, proof theoretically: proof search

The sequent calculus FLewc

Multiplicative rules

$$\frac{}{p \Rightarrow p}$$

$$\frac{A, B, X \Rightarrow C}{A \cdot B, X \Rightarrow C} \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X, Y \Rightarrow A \cdot B}$$

$$\frac{A, X \Rightarrow B}{X \Rightarrow A \to B} \qquad \frac{X \Rightarrow A \qquad B, Y \Rightarrow C}{A \to B, X, Y \Rightarrow C}$$

$$\frac{}{\Rightarrow 1} \qquad \frac{X \Rightarrow C}{1, X \Rightarrow C} \qquad \frac{X \Rightarrow}{X \Rightarrow 0} \qquad \frac{}{0 \Rightarrow}$$

Additive rules

$$\frac{A_i, X \Rightarrow C}{A_1 \wedge A_2, X \Rightarrow C} \qquad \frac{X \Rightarrow A \qquad X \Rightarrow B}{X \Rightarrow A \wedge B}$$

$$\frac{A, X \Rightarrow C \qquad B, X \Rightarrow C}{A \vee B, X \Rightarrow C} \qquad \frac{X \Rightarrow A_1}{X \Rightarrow A_1 \vee A_2}$$

Structural rules

$$\frac{X, U, V, Y \Rightarrow C}{X, V, U, Y \Rightarrow C}\,\mathrm{e} \qquad \frac{X \Rightarrow C}{X, U \Rightarrow C}\,\mathrm{w} \qquad \frac{X, U, U \Rightarrow C}{X, U \Rightarrow C}\,\mathrm{c}$$

Intuitionistic decidability and complexity upper bound

Let input formula F have size n. Then: $|\mathrm{subf}(F)| \leq n$.

Define proof search tree rooted at input $\Rightarrow F$

(write down, at each node, the premises of each applicable rule instance)

This is non-terminating (e.g., keep applying c) so need to refine proof search tree construction while retaining completeness:

- Search for minimal proofs (terminate at repeats)
- Logical property: in $X \Rightarrow C$, treat X as set rather than list
- Slightly modify calculus so antecedent is a strictly increasing set

Then every branch in proof search tree has length $\leq n$

Decidability immediate, also PSPACE membership

Kripke's decidability argument for FLec (1959)

Logical property: in $X \Rightarrow C$, we can regard X as a multiset

multiset: record multiplicity of each element (not just membership)

Observation: no explicit contraction rule if the other rules incorporate a fixed amount of contraction

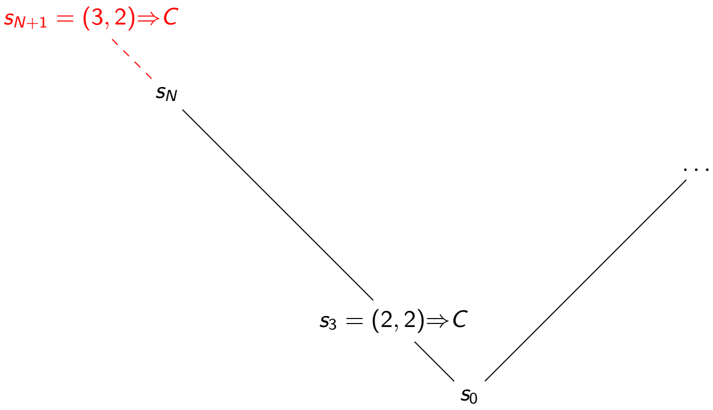
Curry's lemma: modified proof calculus has height-preserving contraction

Search for minimal proofs (Curry's lemma justifies termination at a node if its label s_j can be contracted to label s_i that occurs closer to root)

So a branch $(s_0, s_1, \ldots)$ in the proof search tree is now a sequence of sequents without an increasing pair i.e., $i < j$ implies $s_i \not\leq_c s_j$

Refined proof search: no increasing pair along a branch $(s_0, s_1, \ldots)$

$s_3 \preceq_c s_{N+1}$ so don't write down s_{N+1}



By construction, $(s_0, s_1, s_2, \ldots, s_N)$ satisfies $\forall ij. i < j$ implies $s_i \not\preceq_c s_j$

Is every branch finite? Max length? Enter well-quasi-orders (wqo)

Well-quasi-orders

$\preccurlyeq$ reflexive and transitive binary relation on set X such that

every sequence $(a_0, a_1, a_2, \ldots)$ in X without an increasing pair is finite

Example: $(\mathbb{N}, \leq)$ is a wqo: $7, 5, 4, 2, 0$ and that's it

More generally, wqo extends well-foundedness

More examples of wqos

- $(\mathbb{N}^2, \leq)$ with product order $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$

  So $(2,2) \leq (3,4)$. But $(2,2) \not\leq (3,1)$ and $(3,1) \not\leq (2,2)$.

  Why is $(\mathbb{N}^d, \leq)$ a wqo? Suppose that $(a_i)_{i\in\mathbb{N}}$ does not contain an increasing pair. Then it has subsequence $(a_{r_1(i)})_{i\in\mathbb{N}}$ that is increasing in the first coordinate. The latter has subsequence $(a_{r_2(i)})_{i\in\mathbb{N}}$ that is increasing in the first two coordinates. Ultimately, obtain subsequence increasing in every coordinate, hence we find an increasing pair. Contradiction.

More examples of wqos

- $(\mathbb{N}^2, \leq)$ with product order $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$
- If $(X, \preceq_1)$ and $(Y, \preceq_2)$ are wqos, then $(X \times Y, \prec_{(1,2)})$ is a wqo
- Let Ω be a finite set of formulas. Define the set of sequents

$$S_\Omega = \{X \Rightarrow C \mid X \text{ is multiset in } \Omega \text{ and } C \in \Omega \cup \{\epsilon\}\}$$

Define contraction ordering $X \Rightarrow C \preceq_c Y \Rightarrow C$ iff $Y \Rightarrow C$ can be made $X \Rightarrow C$ using contraction rule

Then $(S_\Omega, \preceq_c)$ is a wqo.

Finiteness of proof search tree in FLec is now evident, hence its decidability.

It will be convenient to ignore succedent and approximate S_Ω as simply $\mathbb{N}^d$ where $d = |\Omega|$, and even ignore that contraction is not exactly the product ordering, and hence regard the wqo as $(\mathbb{N}^d, \leq)$.

Towards complexity: max length of bad sequences

A seq without increasing pair is called a bad sequence. Here is a bad sequence in $(\mathbb{N}, \leq)$:

$$\underbrace{7, 5, 4, 2, 0}_{\text{length}=5}$$

bad sequence length depends on starting element

What is max length of a bad sequence in $(\mathbb{N}^2, \leq)$ starting at $(2, 2)$?

example 1: $(2,2),(4,1),(3,1),(2,1),(1,1),(0,1),(0,0)$

example 2: $(2,2),(4000,1),(3999,1),(3998,1),\ldots$

Finite but no max length

reason: arbitrary jumps like $2 \mapsto 4000$

Controlled bad sequences

Restrict magnitude of jumps using a control function g

$$\underbrace{a_0}_{\|a_0\| \leq t} \quad a_1 \quad \ldots \quad \underbrace{a_i}_{\|a_i\| \leq g^i(n)} \quad \ldots \quad a_{L^g_{\mathcal{W}}(\|a_0\|)}$$

$\Big($ control function $g : \mathbb{N} \to \mathbb{N}$ is monotone with $g(n) \geq n$

$\|a_0\| \leq t \qquad \|a_i\| \leq g^i(n) \qquad \{a \in X \mid \|a\| \leq n\}$ is finite for every $n\Big)$

König's lemma: every (g, t)-controlled bad sequence has maximum length

(Consider enumeration tree of all (g, t)-controlled bad sequences)

$L^g_{\mathcal{W}} : \mathbb{N} \to \mathbb{N}$ assigns, to each $t \in \mathbb{N}$, the maximum length of a (g, t)-controlled bad sequence

Controlled bad sequences

Restrict magnitude of jumps using a control function g

$$\underbrace{a_0}_{\|a_0\|\leq t} \quad a_1 \quad \ldots \quad \underbrace{a_i}_{\|a_i\|\leq g^i(n)} \quad \ldots \quad a_{L^g_{\mathcal{W}}(\|a_0\|)}$$

For $(\mathbb{N}^d, \leq)$ it depends on control function g, starting size t, and d

Removing d, and with g prim rec: upper bounded by Ackermann function (McAloon 1984, Figueira Figueira Schmitz Schnoebelen 2011)

Orderings over $\mathcal{P}_f(\mathbb{N}^d)$: length function upper bounded by hyper-Ackermann function (Balasubramanian 2020)

$X \leq_{\text{maj}} Y$ iff $\forall x \in X.\exists y \in Y.x \leq y$ $\quad X \leq_{\text{min}} Y$ iff $\forall y \in Y.\exists x \in X.x \leq y$

Fast-growing complexity classes (Schmitz)

F_ω an Ackermannian function closed under primrec functions

F_{ω^ω} a hyper-Ackermannian function closed under Ackermannian functions

FLec is $\mathbf{F}_\omega$-complete (Urquhart, 1999)

From FLec to its axiomatic extensions

Observation: the arguments are not sensitive to the form of the proof rules. However, we do need subformula property (analyticity) as it ensures fixed dimension for the wqo

Many axiomatic extensions of FL have analytic (hyper)sequent calculi (Ciabattoni Galatos Terui 2008) based on substructural hierarchy of axioms

hypersequent = multiset of sequents

$$\frac{h \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \qquad\qquad h \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{h \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 | \Delta_1, \Delta_2 \Rightarrow \Pi_2}\ \text{com}$$

Theorem (RR 2020)

Every hypersequent calculus extension of FLec *is decidable.*

Theorem (Balasubramanian, Lang, RR 2021)

Every hypersequent calculus extension of FLec *and* FLew *is in F_{ω^ω}*

Prominent fuzzy logic MTL is in $\mathbf{F}_{\omega^\omega}$.

Argument for extensions of FLew uses forward proof search

Further results

Contraction, weakening generalise to knotted axioms $x^n \to x^m$ for $n > 0$

Exchange can be replaced by weaker forms of commutativity

$$xy_1xy_2 \cdots y_kx \leftrightarrow x^{a_0}y_1x^{a_1}y_2 \cdots y_kx^{a_k} \text{ with } a_0 + a_1 + \cdots + a_k = k + 1$$

For these generalisations, we develop corresponding wqos and length theorems to get decidability and upper bounds

Also, complexity of deducibility problem (is F deducible from finite set Γ?)

We also obtain lower bounds using algebraic counter machines, extending work by Galatos and St John (2022), reduction from Urquhart's EACMs

Upper bounds: axiomatic extensions with cut-free (hyper)sequent calculus

Theorem (Greati, RR 2024)

FLw *is* $\mathbf{F}_{\omega^\omega}$*-complete.*

Theorem (Galatos, Greati, RR, St John (in preparation))

The complexity of deducibility in an extension of FL *by (weak) commutativity and a non-integral knotted axiom is* $\mathbf{F}_\omega$*-complete.*
Deducibility for extensions with sequent axioms is in $\mathbf{F}_\omega$*, and with hypersequent axioms is in* $\mathbf{F}_{\omega^\omega}$*.*

	Logic(s)	Provability			Deducibility		
		Decidability	LB	UB	Decidability	LB	UB
Base logics	$\mathbf{FL_e}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	N[93]	–	–
	$\mathbf{FL_{ew}}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	FEP[52] PS(6.5)[b]	TOWER [96]	TOWER [96]
	$\mathbf{FL_{ec}}$	FMP[113] PS[115]	$\mathbf{F}_\omega$[58]	$\mathbf{F}_\omega$[58]	FEP[52] PS[115][a]	$\mathbf{F}_\omega$[58]	$\mathbf{F}_\omega$[58][a]
	$\mathbf{FL}_{\mathbf{ec}(m,1)}, m > 2$	FEP[52] PS[53]	$\mathbf{F}_\omega$(10.4)[a]	$\mathbf{F}_\omega$(5.20)	FEP[52] PS[53]	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL}_{\mathbf{ec}(m,n)}, n \geq 2$	FEP[52] PS(5.19)	$\mathbf{F}_\omega$(10.4)[a]	$\mathbf{F}_\omega$(5.20)	FEP[52] PS(5.19)	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL}_{\mathbf{e}(\vec{a})\mathbf{c}(m,n)}$	FEP[56] PS(7.30)	PSPACE [54]	$\mathbf{F}_\omega$(7.31)	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(7.31)
	$\mathbf{FL}_{\mathbf{ew}(1,n)}, n \geq 2$	FEP[52] PS[53]	PSPACE [54]	PSPACE [54]	FEP[52] PS(6.5)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL}_{\mathbf{ew}(m,n)}, m \geq 2$	FEP[52] PS(6.5)	PSPACE [54]	$\mathbf{F}_\omega$(6.14)	FEP[52] PS(6.5)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL}_{\mathbf{e}(\vec{a})\mathbf{w}(m,n)}$	FEP[56] PS(7.33)	PSPACE [54]	$\mathbf{F}_\omega$(7.35)	FEP[56] PS(7.33)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(7.35)
	$\mathbf{FL_i}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	FEP[105] PS[59]	$\mathbf{F}_{\omega^\omega}$[59]	$\mathbf{F}_{\omega^\omega}$[59]
	$\mathbf{FL}_{\mathbf{c}(m,n)}$	N[60]	–	–	N[94]	–	–
	$\mathbf{FL}_{\mathbf{w}(1,2)}$	FMP[67] PS[54]	PSPACE [54]	PSPACE [54]	FEP[107]	open	open
	$\mathbf{FL}_{\mathbf{w}(1,n)}$	FMP[67] PS[54]	PSPACE [54]	PSPACE [54]	open	open	open
	$\mathbf{FL}_{\mathbf{w}(m,n)}, m > 1$	open	PSPACE [54]	open	N[94]	–	–
$\mathcal{A} \subseteq \mathcal{N}_2$	$\mathbf{FL_{ec}}(\mathcal{A})$	FEP[56] PS[66]	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$[65]	FEP[56] PS[65][a]	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$[65][a]
	$\mathbf{FL}_{\mathbf{ec}(m,n)}(\mathcal{A})$	FEP[56] PS(5.19)	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$(5.20)	FEP[56] PS(5.19)	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL}_{\mathbf{e}(\vec{a})\mathbf{c}(m,n)}(\mathcal{A})$	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$(7.31)	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$(7.31)
	$\mathbf{FL_{ew}}(\mathcal{A})$	FEP[56] PS[65]	PSPACE [54]	$\mathbf{F}_\omega$[65]	FEP[56] PS(6.5)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$[65][a]
	$\mathbf{FL}_{\mathbf{ew}(m,n)}(\mathcal{A})$	FEP[56] PS(6.5)	PSPACE [54]	$\mathbf{F}_\omega$(6.14)	FEP[56] PS(6.5)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL}_{\mathbf{e}(\vec{a})\mathbf{w}(m,n)}(\mathcal{A})$	FEP[56] PS(7.33)	PSPACE [54]	$\mathbf{F}_\omega$(7.35)	FEP[56] PS(7.33)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$(7.35)
	$\mathbf{FL_i}(\mathcal{A})$	FEP[67] PS[59]	PSPACE [54]	$\mathbf{F}_{\omega^\omega}$[59]	FEP[67] PS[59]	PSPACE [54]	$\mathbf{F}_{\omega^\omega}$[59]
$\mathcal{A} \subseteq \mathcal{P}_3^\flat$	$\mathbf{FL_{ec}}(\mathcal{A})$	FEP(3.3) PS[65]	open	$\mathbf{F}_{\omega^\omega}$[65]	FEP(3.3) PS[65][a]	open	$\mathbf{F}_{\omega^\omega}$[65][a]
	$\mathbf{FL}_{\mathbf{ec}(m,n)}(\mathcal{A})$	FEP(3.3) PS(5.19)	open	$\mathbf{F}_{\omega^\omega}$(5.20)	FEP(3.3) PS(5.19)	open	$\mathbf{F}_{\omega^\omega}$(5.20)
	$\mathbf{FL}_{\mathbf{e}(\vec{a})\mathbf{c}(m,n)}(\mathcal{A})$	FEP(3.3) PS(7.30)	open	$\mathbf{F}_{\omega^\omega}$(7.31)	FEP(3.3) PS(7.30)	open	$\mathbf{F}_{\omega^\omega}$(7.31)
	$\mathbf{FL_{ew}}(\mathcal{A})$	FEP(3.3) PS[65]	open	$\mathbf{F}_{\omega^\omega}$[65]	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$[65][a]
	$\mathbf{FL}_{\mathbf{ew}(m,n)}(\mathcal{A})$	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$(6.14)	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$(6.14)
	$\mathbf{FL}_{\mathbf{e}(\vec{a})\mathbf{w}(m,n)}(\mathcal{A})$	FEP(3.3) PS(7.33)	open	$\mathbf{F}_{\omega^\omega}$(7.35)	FEP(3.3) PS(7.33)	open	$\mathbf{F}_{\omega^\omega}$(7.35)
	$\mathbf{FL_i}(\mathcal{A})$	FEP(3.3) PS(8.13)	open	$\mathbf{F}_{\omega^{\omega^{\omega^\omega}}}$(8.20)	FEP(3.3) PS(8.13)	open	$\mathbf{F}_{\omega^{\omega^{\omega^\omega}}}$(8.20)

Part II: Tighter upper bounds for fragments

The lower bounding argument for FLec makes crucial use of the lattice connectives

So what is the complexity of the multiplicative fragment Lec?

Schmitz 2016: Lec is 2EXPTIME-complete

Argument via automata-theoretic reductions using crucial result by (Demri *et al* 2013)

Lazić & Schmitz 2021: new algebraic upper bounding argument for suitable transition systems

Collaboration with Amir Akbar Tabatabai and Adrián Puerto Aubel:

- Axiomatise what is needed to make that argument work
- Abstract the idea of infinitely many copies of a formula in a sequent
- Extend to further logics

Aim here: informal motivation for the reliance on multiplicative rules, introduce our abstraction

Lazić & Schmitz's argument specified to Lec

Input: sequent $s \in$ Lec. Let's ignore succedent and consider $s \in \mathbb{N}^d$

Define $\mathcal{D}_k$ as the collection of finite sets from which no element above s is deducible in at most k steps:

$$\mathcal{P}_f(\mathbb{N}^d) \supseteq \mathcal{D}_k = \{S \subseteq_f \mathbb{N}^d \mid S \not\vdash^k x \text{ for every } x \geq s\}$$

$\mathcal{P}_f(\mathbb{N}^d \setminus \uparrow s) = \mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \ldots$

Recall: $(\mathcal{P}_f(\mathbb{N}^d), \leq_{\mathrm{maj}})$ is a wqo. Here $X \leq_{\mathrm{maj}} V$ iff $\forall x \in X \exists y \in Y. x \leq y$

Also, each $\mathcal{D}_i$ can be checked to be downset wrt $\leq_{\mathrm{maj}}$. This implies stabilisation at some step $L(s)$:

$\mathcal{D}_0 \supset \mathcal{D}_1 \supset \mathcal{D}_2 \supset \ldots \supset \mathcal{D}_{L(s)} = \mathcal{D}_{L(s)+1}$

The idea is to give a sharper (primrec) bound on $L(s)$ utilising the fact that the proof calculus is Lec (multiplicative)

Measuring $\{\mathcal{D}_i\}_{i=0}^{L(s)}$ through decomposition as union of ideals

It can be shown that $\mathcal{D}_k = \cup_i \mathcal{P}_f(D_{k,i})$ for downsets $D_{k,i} \subseteq \mathbb{N}^d$

$D \subseteq \mathbb{N}^d$ downset has a unique decomposition as $D = \cup_{j=1}^{s} I_j$ such that $I = \downarrow(a_1^i, \ldots, a_d^i)$ and each $a_r^i \in \mathbb{N} \cup \{\omega\}$

Let $I = \downarrow(a_1, \ldots, a_d)$. We can measure the extent to which I is (in)finite:

$$|I| := \max\{a_i \mid a_i < \omega\} \text{ (its finite-size is largest finite coordinate)}$$
$$\omega(I) := \{i \mid a_i = \omega\} \text{ (its infinite-size is set of infinite coordinates)}$$

For downset $D = \cup_{j=1}^{s} I_j \subseteq \mathbb{N}$, define $|D| = \max_{j=1}^{s} |I_j|$

For downset $\mathcal{D} = \cup_i \mathcal{P}_f(D_i) \subseteq \mathcal{P}_f(\mathbb{N}^d)$, define $|\mathcal{D}| = \max_i |D_i|$

It can be shown: for $0 \leq i \leq L(s)$: $|\mathcal{D}_i| \leq g^i(|\mathcal{D}_0|)$

Next: transfer this size control to a suitable chain of ideals on $\mathbb{N}^d$

Selecting a sequence $\{I_j\}_{j=0}^{L(s)-2}$

$$
\begin{array}{ccccc}
\mathcal{D}_0 & \supset & \mathcal{D}_k & \supset & \mathcal{D}_{L(s)} \\
 & & \cup\!\mid & & \\
 & & \mathcal{P}_f(D_k) & &
\end{array}
$$

$$
\begin{array}{ccccc}
D_0 & \supset & D_k & \supset & D_{L(s)-1} \\
 & & \cup\!\mid & & \\
 & & I_k & &
\end{array}
$$

By suitable choice of $D_0 \supset D_1 \supset \ldots \supset D_{L(s)-1}$, identify $\{I_j\}_{j=0}^{L(s)-2}$ such that

(i) I_k is a maximal ideal in D_k but not in D_{k+1} for $k \leq L(s) - 3$

(ii) $\omega(I_k) \supseteq \omega(I_{k+1})$ (set of infinite coordinates decreases in $\leq d$ steps)

(iii) for $i < j \leq L(s) - 2$, if $\omega(I_i) = \omega(I_j)$ then $|I_j| \leq g^i(|\mathcal{D}_0|)$
(at a plateau, the ideals do not increase in finite-size; this means that we can bound the length $c(\omega(I_i), g^i(|\mathcal{D}_0|))$ of plateau by enumerating)

This is enough to upper bound $L(s)$ by counting

Selecting a sequence $\{I_j\}_{j=0}^{L(s)-2}$ (cont)

By suitable choice of $D_0 \supset D_1 \supset \ldots \supset D_{L(s)-1}$, identify $\{I_j\}_{j=0}^{L(s)-2}$ such that

(i) I_k is a maximal ideal in D_k but $I \not\subseteq D_{k+1}$

(ii) $\omega(I_k) \supseteq \omega(I_{k+1})$

(iii) for $i < j \leq L(s) - 2$, if $\omega(I_i) = \omega(I_j)$ then $|I_j| \leq g^i(|\mathcal{D}_0|)$

Let us motivate how to obtain these properties.

Identify rule instance with premises in D_{k+1} and conclusion in $D_k \setminus D_{k+1}$

We need to relate with ideals, since that is where we have some control

Lift rule instance to ideals, relying on multiplicativity (the point is that a premise can be increased independently and still be a legal rule instance)

$$\frac{x \Rightarrow C \qquad y \Rightarrow D}{x, y \Rightarrow C.D} \qquad \frac{\uparrow x \Rightarrow C \qquad y \Rightarrow D}{\uparrow x, y \Rightarrow C.D}$$

Notation. $\hat{S} := S \cup \{x \mid S \vdash^1 x\}$ (sequents provable in a single step)

Abstract version of Lazić & Schmitz's tighter upper bound argument

Definition

An *measured wqo* $(X, \leq, \omega, |\cdot|)$ is defined

1. $(X, \leq)$ is a wqo, and $|\cdot| : Id(X, \leq) \to \mathbb{N}$ is a norm function on $(X, \leq)$
2. $\omega : Id(X, \leq) \to (V, \preceq)$ where $(V, \preceq)$ is a poset, and
   (i) monotonicity: $I \subseteq J$ implies $\omega(I) \preceq \omega(J)$
   (ii) complement: if $I \subseteq J$ and $\omega(I) = \omega(J)$, then $|I| \leq |J|$

Definition

A *measured proof system* $(X, \leq, \omega, |\cdot|, P, g)$ is defined

1. $(X, \leq, \omega, |\cdot|)$ is measured wqo, P proof system over X, g control function
2. $x \leq y$ then $\{y\} \vdash x$ (proof-theoretic admissibility of the contraction rule)
3. $(S, y) \in P$ and $I \cap S \neq \emptyset$, then $\exists J \in Id(X, \leq)$ s.t. $J \subseteq \downarrow \widehat{S \cup I}$, and $y \in J$, $\omega(I) \preceq \omega(J)$
4. ...

Theorem (Akbar Tabatabai, Puerto Aubel, RR (in preparation))

Let $(X, \leq, \omega, |\cdot|, P, g)$ be a measured proof system. $n := |\mathcal{D}_0|$, $\Omega \prec \Omega' \prec \ldots$

$$L(s) \leq 2 + c(\Omega, n) + c(\Omega', g^{c(\Omega,n)}(n)) + c(\Omega'', g^{c(\Omega', g^{c(\Omega,n)})}(n)) + \ldots + s(\Omega^{'\cdots'}, \ldots)$$

Open problems

Decidability of $\mathsf{FLe} + x \to x^2 \vee 1$ is open: it seems beyond reach of the (wqo admissibility, minimal proofs) technology

$$\frac{X, X, Z \Rightarrow C \qquad Z \Rightarrow C}{X, Z \Rightarrow C}$$

Contraction uses backward proof search, weakening uses forward. Can we unify?

Building on methods by Blok, van Alten, Galatos, Jipsen and others, we obtain algebraic counter models via FEP. How to bound the size of these models?

No non-trivial lower bound known for MTL ($\mathsf{FLew} + (p \to q) \vee (q \to p)$)

More generally, how to extend lower bounding methods to hypersequent calculi?