Quantitative Graded Semantics and Modal Logics

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Introduction: Behavioural Metrics

- Behavioural metrics offer fine-grained notion of process comparison
  - Fuzzy, weighted, metric transition systems
  - Markov chains, Markov decision processes

- Flag low distance instead of just inequivalence

- Arranged on linear-time / branching-time spectrum
  e.g. on metric LTS (Fahrenberg/Legay/Thrane 2011)

  - (ready) simulation, traces, failures etc.

- Here: generic framework for spectra of behavioural metrics

  - based on coalgebra and graded monads (graded semantics)
Introduction: Characteristic Logics

Classically, a modal logic is characteristic for a given behavioural equivalence if

$$\text{behavioural equivalence} = \text{logical indistinguishability}$$

$$\rightarrow$$ modal formulae witness inequivalence

A quantitative modal logic is characteristic for a given behavioural metric if

$$\text{behavioural distance} = \text{logical distance}$$

$$\rightarrow$$ modal formulae witness high distance
Classically, characteristic modal logics are compositional fragments of branching-time logics.

- E.g. trace equivalence: $\Diamond_a$, $\top$ ($,$ $\lor$)

Negative result: Trace distance on probabilistic metric transition systems has no characteristic modal logic that is a compositional fragment of a branching-time logic.
Introduction: Graded Quantitative Logics

- **Graded logics**: Canonical notion of logic for graded semantics

- Fragments of branching-time coalgebraic modal logics

- **Invariance** (logical $\leq$ behavioural distance) holds without restriction

- **Expressiveness** (logical $\geq$ behavioural distance) holds under separation

- **Positive examples**: Metric traces (Beohar, Gurke, König, Messing 2023), fuzzy metric traces (new)
Behavioural metrics

E.g. probabilistic systems:

\[ \begin{array}{c}
    a \quad \bullet \quad b \\
    \downarrow 0.5 \quad \downarrow 0.5 \\
\end{array} \quad \begin{array}{c}
    a \quad \bullet \quad b \\
    \downarrow 0.4 \quad \downarrow 0.6 \\
\end{array} \]

– not bisimilar, but „close“

▶ Behavioural distance 0.1 under standard definitions

(earth movers metric)
Coalgebra

Coalgebras = generic reactive systems

- Set $X$ of states
- Transition structure $X \rightarrow FX$
- Functor $F$ is the type of the system.
- E.g. $F = \mathcal{P}$: Non-deterministic branching
Metric Labelled Systems as Coalgebras

- Metric transition systems, $L$ metric space of labels:

$$X \rightarrow \mathcal{P}(L \times X)$$

- Probabilistic metric transition systems

$$X \rightarrow \mathcal{D}(L \times X)$$

- Fuzzy metric transition systems:

$$X \rightarrow \mathcal{F}(L \times X)$$

$\mathcal{F}X = [0, 1]^X$ fuzzy powerset
Trace Distances

- Metric transition system $X \to \mathcal{P}(L \times X)$:
  - have sup metric on traces
  - form crisp trace sets
  - measure Hausdorff distance

- Fuzzy transition system $X \to \mathcal{F}(L \times X)$:
  - form fuzzy trace sets
  - measure fuzzy Hausdorff distance

- Probabilistic metric transition system $X \to \mathcal{D}(L \times X)$:
  - have Manhattan metric on traces
  - form length-$n$ trace distributions
  - measure Kantorovich distance
Recall: Monads and Theories

(Algebraic) theories \((\Sigma, E)\) consist of

- (algebraic) signature \(\Sigma\) – operations with arities
- equations \(E\).

Correspond to monads \(M\) (on \(\mathbf{Set}\)); on set \(X\):

- \(MX = \Sigma\)-Terms with variables in \(X\) / equations
- \(\eta : X \rightarrow MX\) variables-as-terms (unit)
- \(\mu : MMX \rightarrow MX\) substitution (multiplication)
Graded Monads and Theories

(Smirnoff 2008)

Graded theories \((\Sigma, d, E)\) consist of

- \(d : \Sigma \rightarrow \mathbb{N}\) depth
  - \(\rightarrow\) terms of uniform depth
- equations \(E\) of uniform depth

Correspond to graded monads \((M_n)_{n<\omega}\):

- \(M_nX = \Sigma\)-terms of uniform depth \(n\) over \(X\)
- \(\eta : X \rightarrow M_0X\)
- \(\mu^{nk} : M_nM_kX \rightarrow M_{n+k}X\)
Graded Semantics

of $G$-coalgebras = graded monad $(M_n) +$ natural transformation

$$\alpha_X : GX \rightarrow M_1 X$$

▶ Inductively defined sequence

$$\gamma^{(0)} : X \xrightarrow{M_0! \circ \eta} M_0 1 \quad \gamma^{(n+1)} : X \xrightarrow{\alpha \circ \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu^{1n}} M_{n+1} 1$$

of $n$-step behaviour maps $\gamma^{(n)} : X \rightarrow M_n 1$
Use functor $G$ on metric spaces (e.g. lift a set functor)

Map into graded monad $(M_n)$ on metric spaces

Graded behavioural distance on $\gamma: X \to GX$:

$$d(x, y) = \bigvee_{n<\omega} d(\gamma^{(n)}(x), \gamma^{(n)}(y))$$
Graded Quantitative Equational Theories

Graded version of quantitative equational theories
(Mardare/Panangaden/Plotkin 2016)

▶ $\approx_\varepsilon$: Equality up to $\varepsilon$ (quantitative equality)

▶ Axioms of the form

$$\Gamma \vdash s \approx_\varepsilon t$$

with $\Gamma$ set of quantitative equalities on the variables

▶ Expected rules including triangle inequality and non-expansiveness of all operators
Branching time

Metric traces:

Depth 0: Quantitative join semilattices
( = join semilattices + non-expansiveness of join)
→ \( M_0 = \mathcal{P}_\omega \) with Hausdorff distance

Depth 1: Operations \( a(\cdot) \) for \( a \in L \), axioms

\[
\vdash a(0) =_0 0 \quad \vdash a(x + y) =_0 a(x) + a(y) \\
\quad x = \epsilon \ y \vdash a(x) =_{\max\{\epsilon, d_L(a,b)\}} b(y)
\]

So \( M_n X = \mathcal{P}_\omega (L^n \times X) \), \( L^n \) with product (sup) distance, \( \mathcal{P}_\omega \) with Hausdorff
Fuzzy traces:

Depth 0: Quantitative join semilattices with action of $([0, 1], \land)$,

\[ x =_{\varepsilon} y \vdash r(x) =_{\varepsilon} s(y) \quad \text{when} \quad |r - s| \leq \varepsilon \]

Depth 1: Operations $a(-)$ for $a \in L$, usual trace equations plus

\[ a(r(x)) = r(a(x)) \quad \text{for} \quad r \in [0, 1], \ a \in L \]

so $M_nX = F_\omega(L^n \times X)$ with fuzzy Hausdorff distance

Probabilistic traces: $M_nX = D_\omega(L^n \times X)$, $L^n \times X$ with Manhattan distance, $D_\omega$ with Kantorovich.
Quantitative Coalgebraic Modal Logic

Parametrized over

- Set $\Theta$ of truth constants
- Set $\mathcal{O}$ of propositional operators
- Set $\Lambda$ of modalities

Semantics over $G$-coalgebra $\gamma : X \to GX$:

- Space $\Omega = [0, 1]$ of truth values
- $c$ truth constant: $\hat{c} : 1 \to \Omega$
- $p$ propositional operator: $[[p]] : \Omega^n \to \Omega$ non-expansive
- $L \in \Lambda$: $[[L]] : G\Omega \to \Omega$ non-expansive

\[
[[L\phi]]_{\gamma} = (X \xrightarrow{\gamma} GX \xrightarrow{G[[\phi]]_{\gamma}} G\Omega \xrightarrow{[[L]]} \Omega)
\]

- $[[\phi]] : X \to \Omega$ invariant, i.e. non-expansive w.r.t. behavioural distance
**Theorem**  There is no characteristic quantitative (coalgebraic) modal logic with unary modalities for trace distance on probabilistic metric transition systems.

**Proof**  Assume $\mathcal{L}$ is invariant.

- Show that modalities $\llbracket L \rrbracket : \mathcal{D}_\omega(L \times \Omega) \to \Omega$ are affine.

- In

  
  ![Diagram](image_url)

  with $d(a, b) = \nu < 1$, $x$ and $y$ have behavioural distance $\nu$ but logical distance $\leq \nu^2$. 

\[\square\]
Quantitative Graded Logics

Given graded semantics $\alpha, (M_n)$, a logic $\mathcal{L} = (\Theta, \mathcal{O}, \Lambda)$ is graded if

- $\Omega$ $M_0$-algebra
- propositional operators $\llbracket p \rrbracket : \Omega^n \to \Omega$ homomorphic
- modalities $\llbracket L \rrbracket : GX \to \Omega$ factor through $M_1$-algebras

$$\llbracket L \rrbracket : M_1\Omega \to \Omega$$

**Theorem** Uniform-depth formulae in a graded logic $\mathcal{L}$ are invariant under the graded semantics
Quantitative Graded Logics: Examples

► Metric traces:
  ▶ Propositional operators: e.g. none, or joins
  ▶ Modalities: $\Diamond_a, \ a \in L$,

  \[
  \sem{\Diamond_a}(f)(U) = \bigvee_{(b,x) \in U} (1 - d(a,b)) \land f(x) \text{ for } f : X \to [0,1], U \in \mathcal{P}_\omega(L \times X)
  \]

► Fuzzy metric traces:
  ▶ Propositional operators: e.g. none, or joins
  ▶ Modalities $\Diamond^c_a, \ a \in L, \ c \in [0,1]$,

  \[
  \sem{\Diamond^c_a}(f)(A) = \bigvee_{x \in X} A(a, x) \land f(x) \land (c - d(a,b)) \text{ for } f : X \to [0,1], A \in \mathcal{F}_\omega(L \times X)
  \]

► Probabilistic traces:
  ▶ Propositional operators: e.g. none, or affine combinations
  ▶ Modalities $\Diamond_a, \ a \in L$,

  \[
  \sem{\Diamond_a}(f)(\mu) = \sum_{x \in X} \mu(a, x) \cdot f(x) \text{ for } f : X \to [0,1], \mu \in \mathcal{D}_\omega(L \times X)
  \]
**Expressiveness via Separation**

\( \mathcal{L} \) quantitative graded logic with set \( \Lambda \) of modalities

- Induces \( \mathcal{L} \)-logical distance

Let \( \Phi \) be a condition on cones \( \subseteq \text{Met}(X, [0, 1]) \) that implies *initiality*.

**Theorem.** If \( \mathcal{L} \) is \( \Phi \)-type depth-0-separating and \( \Phi \)-type depth-1 separating, then \( \mathcal{L} \)-logical distance equals graded behavioural distance.

**Definition.** \( \mathcal{L} \) is \( \Phi \)-type depth-1 separating if whenever

\[ \mathcal{A} \subseteq \text{Met}^0(M_n1, [0, 1]) \]

satisfies \( \Phi \) and is closed under the propositional operators, then

\[ \Lambda \mathcal{A} = \{ L(f) \mid L \in \Lambda, f \in \mathcal{A} \} \subseteq \text{Met}^0(M_{n+1}1, [0, 1]) \]

satisfies \( \Phi \).
Positive Examples

▶ The logic of metric streams is expressive (Φ = normed isometry)

▶ The logic of metric traces is expressive (Φ = normed isometry)

▶ The logic of fuzzy metric traces is expressive (Φ = initiality)
Conclusions

- Surprising negative result on existence of characteristic quantitative modal logic for probabilistic metric traces

- The principles of graded characteristic logics carry transfer to the quantitative setting

- Inductive expressiveness criterion becomes more subtle
  - needs strengthened invariant as a parameter

- New positive example:
  Fuzzy metric traces, with slightly subtly choice of modalities
Future Work

- Expressive logic for probabilistic metric traces with higher-arity modalities?

- Game characterization

- Fixpoint characterization

- Existence of expressive sets of modalities?