Quantitative Graded Semantics and Modal Logics

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Introduction: Behavioural Metrics

- Behavioural metrics offer fine-grained notion of process comparison
  - Fuzzy, weighted, metric transition systems
  - Markov chains, Markov decision processes
- Flag low distance instead of just inequivalence
- Arranged on linear-time / branching-time spectrum
  e.g. on metric LTS (Fahrenberg/Legay/Thrane 2011)
  - (ready) simulation, traces, failures etc.
- Here: generic framework for spectra of behavioural metrics
  - based on coalgebra and graded monads (graded semantics)

Introduction: Characteristic Logics

- Classically, a modal logic is characteristic for a given behavioural equivalence if

  $$\text{behavioural equivalence} = \text{logical indistinguishabilty}$$

  $\rightarrow$ modal formulae witness inequivalence

- A quantitative modal logic is characteristic for a given behavioural metric if

  $$\text{behavioural distance} = \text{logical distance}$$

  $\rightarrow$ modal formulae witness high distance

Introduction: Negative Result

- Classically, characteristic modal logics are compositional fragments of branching-time logics
  - E.g. trace equivalence: $\Diamond_a$, $\top$ (, $\vee$)
- **Negative result:** Trace distance on probabilistic metric transition systems has no characteristic modal logic that is a compositional fragment of a branching-time logic.

Introduction: Graded Quantitative Logics

- Graded logics: Canonical notion of logic for graded semantics
- Fragments of branching-time coalgebraic modal logics
- Invariance (logical $\leq$ behavioural distance) holds without restriction
- Expressiveness (logical $\geq$ behavioural distance) holds under separation
- Positive examples: Metric traces (Beohar, Gurke, König, Messing 2023), fuzzy metric traces (new)

Behavioural metrics

E.g. probabilistic systems:

– not bisimilar, but „close“

- Behavioural distance 0.1 under standard definitions (*earth movers metric*)

Coalgebra

Coalgebras = generic reactive systems

- Set X of states
- Transition structure $X \to FX$
- Functor F is the type of the system.
- E.g. $F = \mathcal{P}$: Non-deterministic branching

Metric Labelled Systems as Coalgebras

- Metric transition systems, L metric space of labels:

$$X \to \mathcal{P}(L \times X)$$

- Probabilistic metric transition systems

$$X \to \mathcal{D}(L \times X)$$

- Fuzzy metric transition systems:

$$X \to \mathcal{F}(L \times X)$$

$\mathcal{F}X = [0,1]^X$ fuzzy powerset

Trace Distances

- Metric transition system $X \to \mathcal{P}(L \times X)$:
  - have sup metric on traces
  - form crisp trace sets
  - measure Hausdorff distance
- Fuzzy transition system $X \to \mathcal{F}(L \times X)$:
  - form fuzzy trace sets
  - measure fuzzy Hausdorff distance
- Probabilistic metric transition system $X \to \mathcal{D}(L \times X)$:
  - have Manhattan metric on traces
  - form length-n trace distributions
  - measure Kantorovich distance

Recall: Monads and Theories

(Algebraic) theories (Σ, E) consist of

- (algebraic) signature Σ – operations with arities
- equations E.

Correspond to monads M (on **Set**); on set X:

- $MX = \Sigma$-Terms with variables in X / equations
- $\eta : X \to MX$ variables-as-terms (unit)
- $\mu : MMX \to MX$ substitution (multiplication)

Graded Monads and Theories

(Smirnoff 2008)

Graded theories (Σ, d, E) consist of

- $d : \Sigma \to \mathbb{N}$ depth
  - $\to$ terms of uniform depth
- equations E of uniform depth

Correspond to graded monads $(M_n)_{n<\omega}$:

- $M_n X = \Sigma$-terms of uniform depth n over X
- $\eta : X \to M_0 X$
- $\mu^{nk} : M_n M_k X \to M_{n+k} X$

Graded Semantics

of G-coalgebras = graded monad (M_n) + natural transformation

$$\alpha_X : GX \to M_1 X$$

- Inductively defined sequence

$$\gamma^{(0)} : X \xrightarrow{M_0 ! \circ \eta} M_0 1 \qquad \gamma^{(n+1)} : X \xrightarrow{\alpha \circ \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu^{1n}} M_{n+1} 1$$

of n-step behaviour maps $\gamma^{(n)} : X \to M_n 1$

Quantitative Graded Semantics

- Use functor G on metric spaces (e.g. lift a set functor)
- Map into graded monad (M_n) on metric spaces
- Graded behavioural distance on $\gamma\colon X \to GX$:

$$d(x,y) = \bigvee_{n<\omega} d(\gamma^{(n)}(x), \gamma^{(n)}(y))$$

Graded Quantitative Equational Theories

Graded version of quantitative equational theories
(Mardare/Panangaden/Plotkin 2016)

- $=_\varepsilon$: Equality up to ε (quantitative equality)
- Axioms of the form
$$\Gamma \vdash s =_\varepsilon t$$
with Γ set of quantitative equalities on the variables
- Expected rules including triangle inequality and non-expansiveness of all operators

Graded Quantitative Semantics: Examples (I)

- Branching time
- Metric traces:
  - Depth 0: Quantitative join semilattices
    (= join semilattices + non-expansiveness of join)
    $\to M_0 = \mathcal{P}_\omega$ with Hausdorff distance
  - Depth 1: Operations $a(-)$ for $a \in L$, axioms

$$\vdash a(0) =_0 0 \quad \vdash a(x+y) =_0 a(x) + a(y)$$
$$x =_\varepsilon y \vdash a(x) =_{\max\{\varepsilon, d_L(a,b)\}} b(y)$$

So $M_n X = \mathcal{P}_\omega(L^n \times X)$, L^n with product (sup) distance, $\mathcal{P}_\omega$ with Hausdorff

Graded Quantitative Semantics: Examples (II)

- Fuzzy traces:
  - Depth 0: Quantitative join semilattices with action of $([0,1],\wedge)$,

$$x =_\varepsilon y \vdash r(x) =_\varepsilon s(y) \qquad \text{when } |r-s| \leq \varepsilon$$

  - Depth 1: Operations $a(-)$ for $a \in L$, usual trace equations plus

$$a(r(x)) = r(a(x)) \qquad \text{for } r \in [0,1],\ a \in L$$

    so $M_n X = \mathcal{F}_\omega(L^n \times X)$ with fuzzy Hausdorff distance

- Probabilistic traces: $M_n X = \mathcal{D}_\omega(L^n \times X)$, $L^n \times X$ with Manhattan distance, $\mathcal{D}_\omega$ with Kantorovich.

Quantitative Coalgebraic Modal Logic

Parametrized over

- Set Θ of truth constants
- Set $\mathcal{O}$ of propositional operators
- Set Λ of modalities

Semantics over G-coalgebra $\gamma\colon X \to GX$:

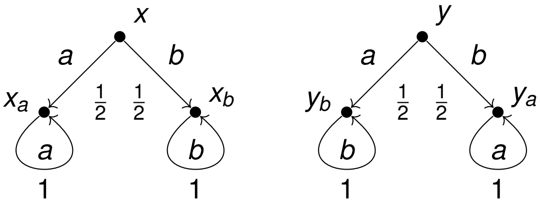
- Space $\Omega = [0, 1]$ of truth values
- c truth constant: $\hat{c}\colon 1 \to \Omega$
- p propositional operator: $[\![p]\!]\colon \Omega^n \to \Omega$ non-expansive
- $L \in \Lambda$: $[\![L]\!]\colon G\Omega \to \Omega$ non-expansive
- $[\![L\phi]\!]_\gamma = (X \xrightarrow{\gamma} GX \xrightarrow{G[\![\phi]\!]_\gamma} G\Omega \xrightarrow{[\![L]\!]} \Omega)$
- $[\![\phi]\!]\colon X \to \Omega$ invariant, i.e. non-expansive w.r.t. behavioural distance

No Characteristic Logic for Probabilistic Metric Traces

Theoren There is no characteristic quantitative (coalgebraic) modal logic with unary modalities for trace distance on probabilistic metric transition systems.

Proof Assume $\mathcal{L}$ is invariant.

- Show that modalities $[\![L]\!]\colon \mathcal{D}_\omega(L \times \Omega) \to \Omega$ are affine.
- In



with $d(a,b) = v < 1$, x and y have behavioural distance v but logical distance $\leq v^2$. □

Quantitative Graded Logics

Given graded semantics $\alpha, (M_n)$, a logic $\mathcal{L} = (\Theta, \mathcal{O}, \Lambda)$ is graded if

- Ω M_0-algebra
- propositional operators $[\![p]\!] \colon \Omega^n \to \Omega$ homomorphic
- modalities $[\![L]\!] \colon GX \to \Omega$ factor through M_1-algebras

$$(\!|L|\!) \colon M_1\Omega \to \Omega$$

Theorem Uniform-depth formulae in a graded logic $\mathcal{L}$ are invariant under the graded semantics

Quantitative Graded Logics: Examples

- Metric traces:
  - Propositional operators: e.g. none, or joins
  - Modalities: $\Diamond_a$, $a \in L$,

$$[\![\Diamond_a]\!](f)(U) = \bigvee_{(b,x)\in U}(1 - d(a,b)) \wedge f(x) \quad \text{for } f\colon X \to [0,1], U \in \mathcal{P}_\omega(L \times X)$$

- Fuzzy metric traces:
  - Propositional operators: e.g. none, or joins
  - Modalities $\Diamond_a^c$, $a \in L$, $c \in [0,1]$

$$[\![\Diamond_a^c]\!](f)(A) = \bigvee_{x\in X} A(a,x) \wedge f(x) \wedge (c - d(a,b)) \qquad \text{for } f\colon X \to [0,1], A \in \mathcal{F}_\omega(L$$

- Probabilistic traces:
  - Propositional operators: e.g. none, or affine combinations
  - Modalities $\Diamond_a$, $a \in L$,

$$[\![\Diamond_a]\!](f)(\mu) = \sum_{x\in X} \mu(a,x) \cdot f(x) \qquad \text{for } f\colon X \to [0,1], \mu \in \mathcal{D}_\omega(L \times X)$$

Expressiveness via Separation

$\mathcal{L}$ quantitative graded logic with set Λ of modalities

- Induces $\mathcal{L}$-logical distance

Let Φ be a condition on cones $\subseteq \mathbf{Met}(X, [0,1])$ that implies initiality.

Theorem. If $\mathcal{L}$ is Φ-type depth-0-separating and Φ-type depth-1 separating, then $\mathcal{L}$-logical distance equals graded behavioural distance

Definition. $\mathcal{L}$ is Φ-type depth-1 separating if whenever $\mathfrak{A} \subseteq \mathbf{Met}^{M_0}(M_n 1, [0,1])$ satisfies Φ and is closed under the propositional operators, then

$$\Lambda\mathfrak{A} = \{L(f) \mid L \in \Lambda, f \in \mathcal{A}\} \subseteq \mathbf{Met}^{M_0}(M_{n+1} 1, [0,1])$$

satisfies Φ.

Positive Examples

- The logic of metric streams is expressive (Φ= normed isometry)
- The logic of metric traces is expressive (Φ= normed isometry)
- The logic of fuzzy metric traces is expressive (Φ= initiality)

Conclusions

- Surprising negative result on existence of characteristic quantitative modal logic for probabilistic metric traces
- The principles of graded characteristic logics carry transfer to the quantitative setting
- Inductive expressiveness criterion becomes more subtle
  - needs strengthened invariant as a parameter
- New positive example:
  Fuzzy metric traces, with slightly subtly choice of modalities

Future Work

- Expressive logic for probabilistic metric traces with higher-arity modalities?
- Game characterization
- Fixpoint characterization
- Existence of expressive sets of modalities?