The LLAMA Seminar
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Overview of Strict Finitistic Logic

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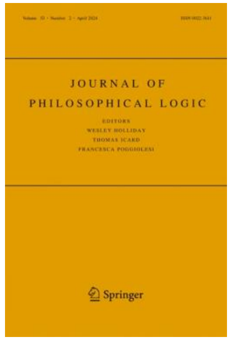
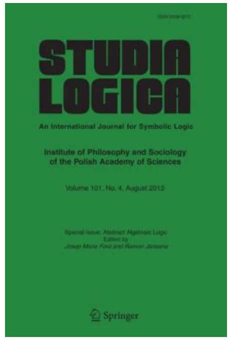
Topics

Strict Finitistic Logics

	Propositional	Quantificational
Without Prevalence		**SFQ**
With Prevalence	$\mathbf{SFP_P}$	$\mathbf{SFQ_P}$



2023: Yamada, calling "**SF**".



2024: Yamada

Topics

What Is Strict Finitism?

- Another constructive view of mathematics.
  - Crispin Wright's semantics: a sketch in 1982.
    - I reconstruct in the classical metatheory.
  - Based on practical constructibility & verifiability.
    - Replaces "in principle" of intuitionism by "in practice".

History

Ideas' Path to Wright

- 1982: Wright, "Strict Finitism"
- 1975: Dummett, "Wang's Paradox"
- 1958: Yessenin-Volpin
- 1935: Bernays

Intuitionism — In principle

Strict Finitism — In practice

Wang's "Paradox"

- The strict finitistic conviction based on constructability in practice.

  1. 0 is practically constructible.
  2. For any practically constructible number, its successor is practically constructible.
  3. There is a number that is not practically constructible. (Finitism)

  ➢ Formalisable in **SFQ**, strict finitistic logic.

The Modality Bridging Principle

- Connects intuitionism (possibility in principle) with strict finitism (possibility in practice).

"A statement is verifiable in principle iff it is verifiable in practice with some finite extension of cognitive resources."

➢ Formalisable in **SFQ_P**, strict finitistic logic with "prevalence".

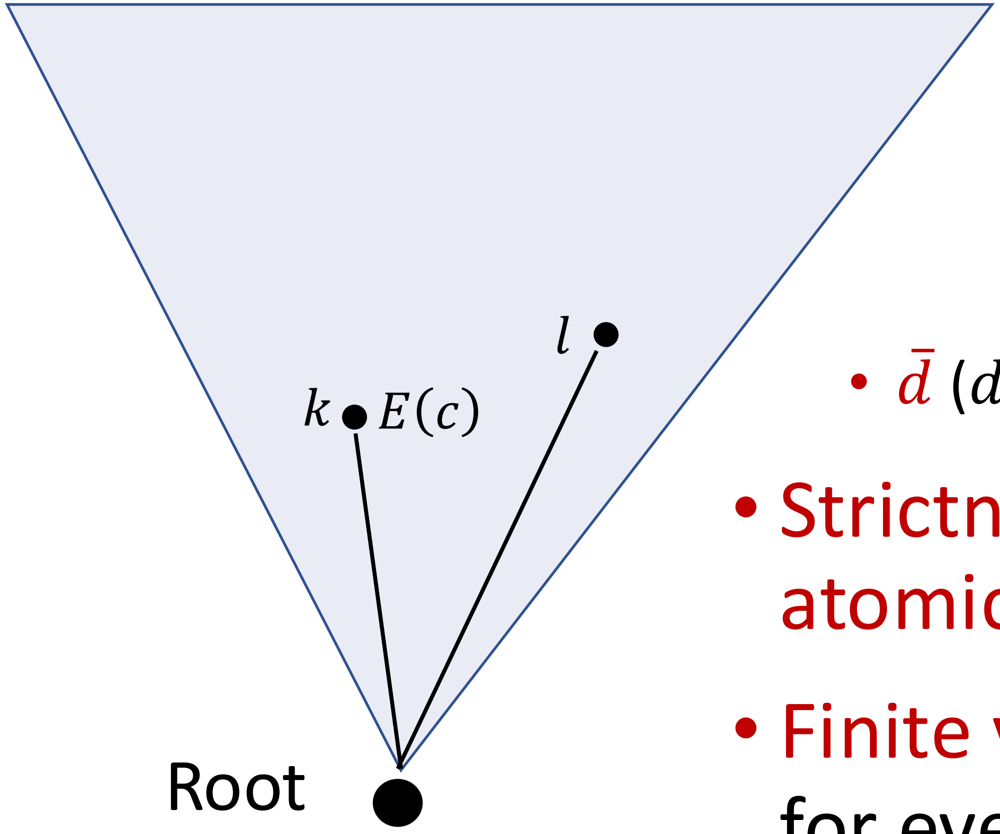
1. Semantics of **SFQ**, the General Logic

Language

- A standard language of first-order predicate logic with…
  - ➢ $\top$ and $\bot$ as atomic formulas.
  - ➢ the existence predicate E for "calculated" or "constructed".
  - ➢ $\neg A$ distinct from $A \to \bot$.
    - ➢ $\sim A \coloneqq A \to \bot$.

$$\text{Term} ::= a|x|f(t)$$
$$\text{Form} ::= \top|\bot|P(t)|A \wedge A|A \vee A|A \to A|\neg A|\forall x A|\exists x A$$

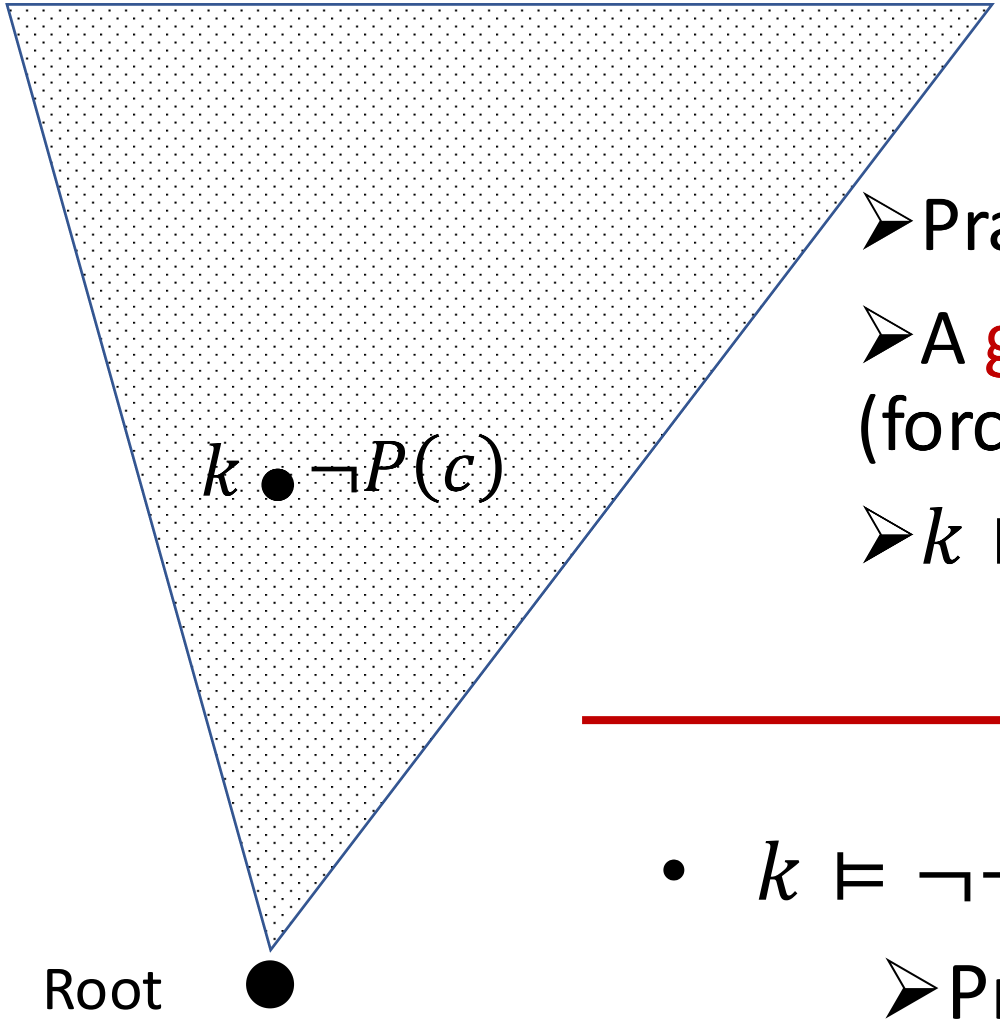
Strict Finitistic Models



- Rooted tree-like intuitionistic model $\langle K, \leq, D, J, v \rangle$ with a constant domain D.
  - $\bar{d}$ $(d \in D)$ is the name of d: $J(\bar{d}) = d$.
- Strictness: $k \vDash P(c)$ implies $k \vDash E(c)$, P atomic.
- Finite verification: $\{P | P^{v(k)} \neq \emptyset\}$ is finite for every $k \in K$.

➢ All possible histories of a human agent's actual verification from our perspective.

“Global” Negation



- $k \vDash \neg A$ iff $l \nvDash A$ for any l.

➢Practically unverifiable.

➢A global formula: if satisfiable, then valid (forced at every node).

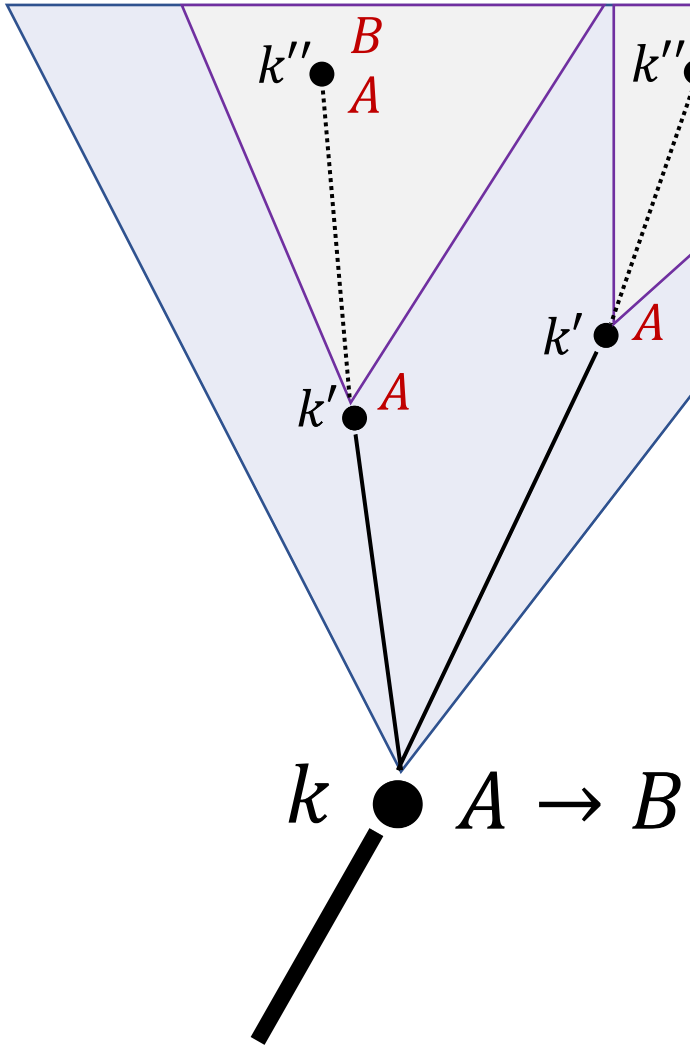
➢$k \vDash \neg P(c)$ does not require $k \vDash E(c)$:

➢Does not require the denotation.

- $k \vDash \neg\neg A$ iff $l \vDash A$ for some l.

➢Practically verifiable.

"Practical" Implication



- $k \vDash A \rightarrow B$ iff for any $k' \geq k$, if $k' \vDash A$, then there is a $k'' \geq k'$ with $k'' \vDash B$.

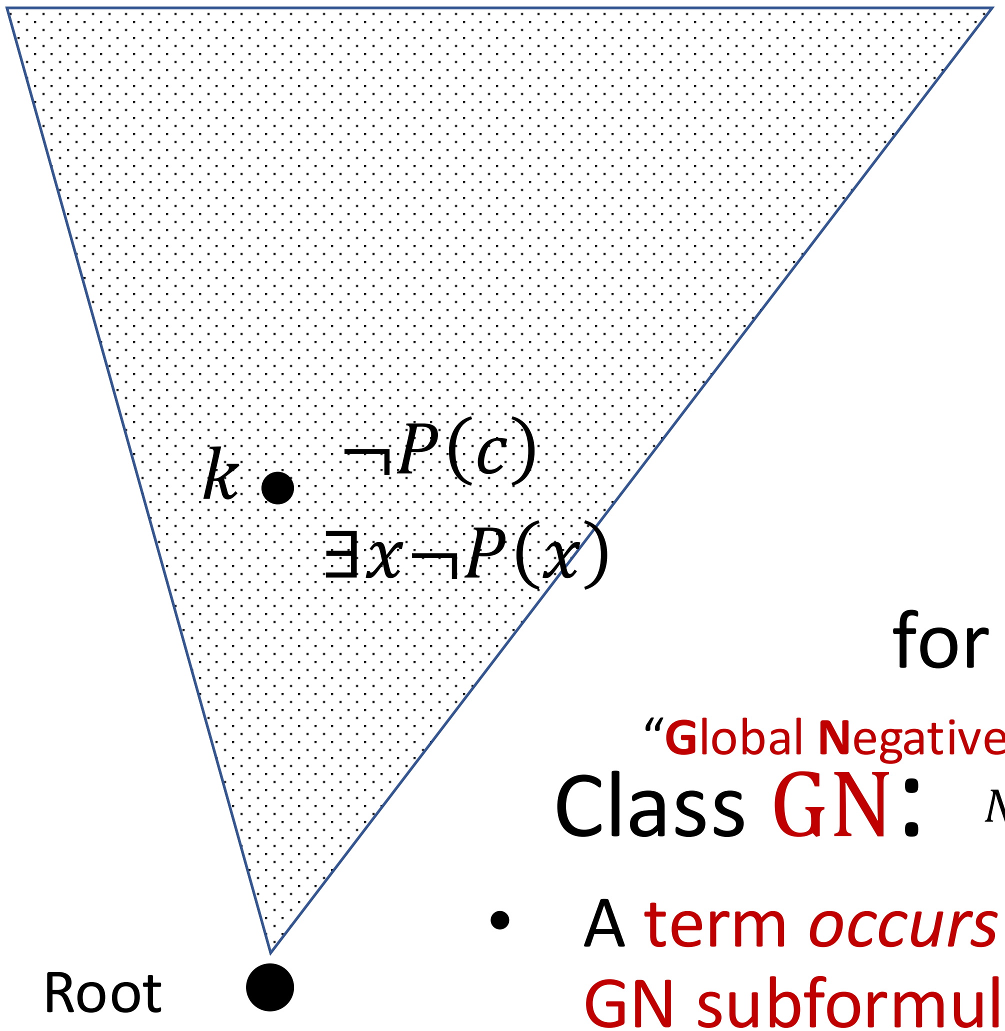
➢ "B comes after A, soon enough".

➢ Intuitionistic implication with time-gap.

- $k \vDash \sim A$ iff for any $k' \geq k$, $k' \nvDash A$.
  - ➢ Intuitionistic, "local" negation.

Ideas

2 Modes of Quantification



- **Local** mode:

  $$P(c) \Longrightarrow \exists x P(x)$$

  for a constructed object.

- **Global** mode:

  $$\underline{\neg P(c)} \Longrightarrow \exists x \underline{\neg P(x)}$$

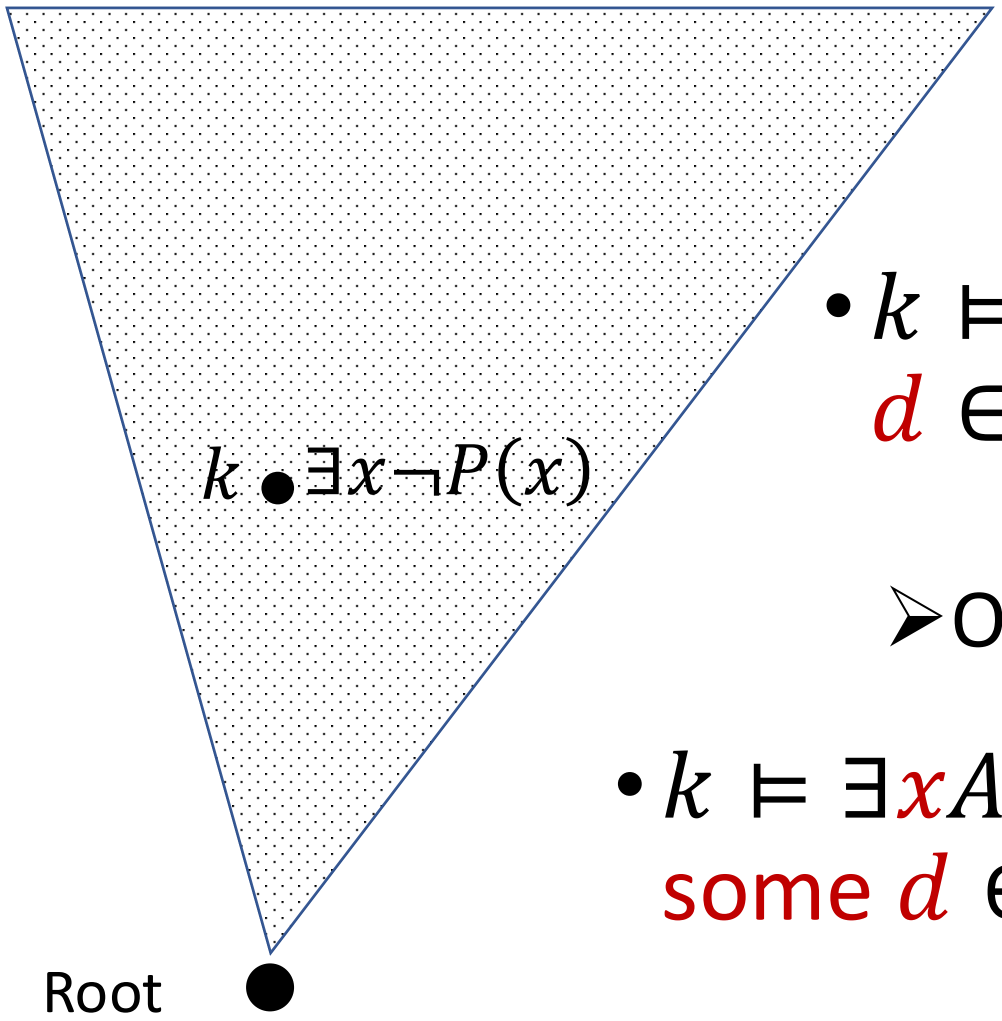
  for an object in the scope of discourse.

"**G**lobal **N**egative"

Class GN: $N ::= \bot \mid \neg \mathrm{Form} \mid N \wedge N \mid N \vee N \mid N \to N \mid \forall x N \mid \exists x N$

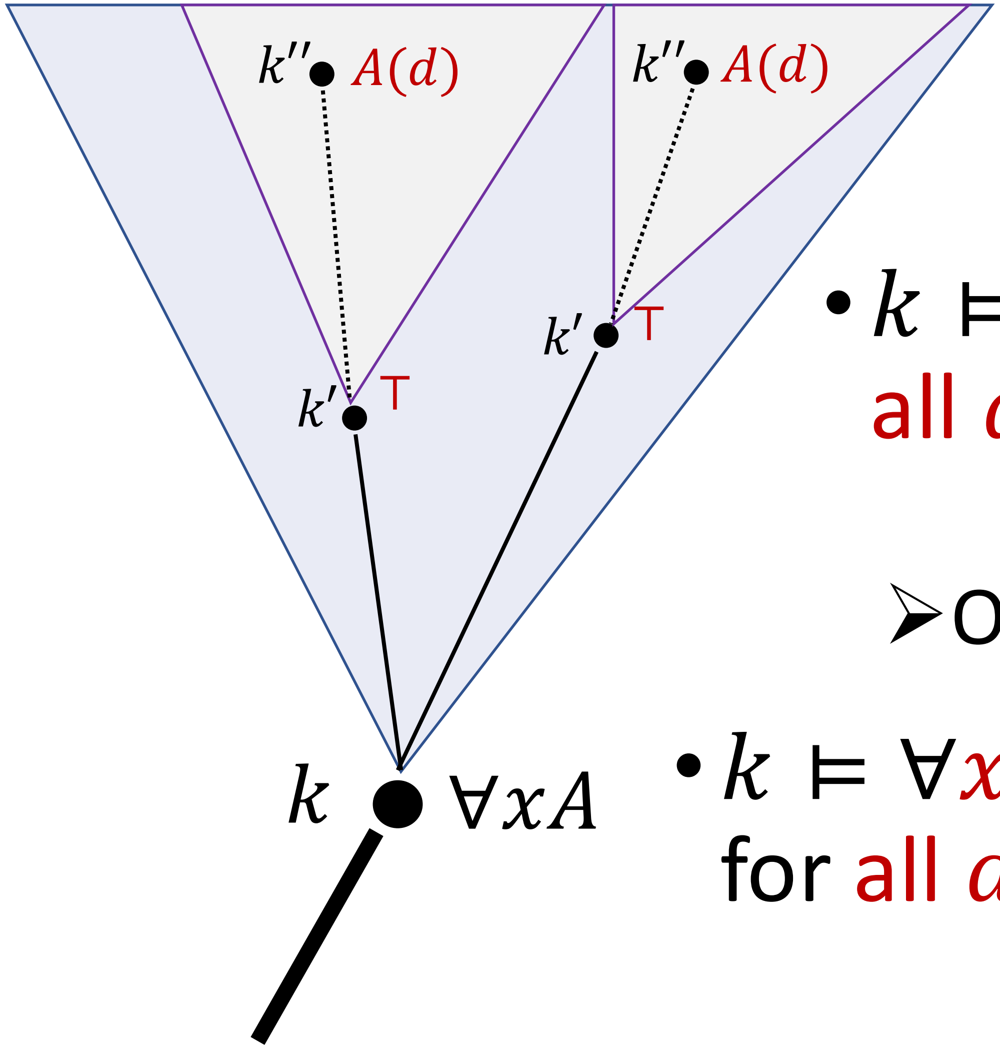
- A term *occurs* in A *globally* if it occurs in a GN subformula of A.

Global & Local ∃



- For x occurring only globally,
  - $k \vDash \exists x A$ iff $k \vDash A[\overline{d}/x]$ for some $d \in D$. (Global quantification)
- Otherwise…
  - $k \vDash \exists x A$ iff $k \vDash E(\overline{d}) \wedge A[\overline{d}/x]$ for some $d \in D$. (Local quantification)

Global & Local ∀



➢ For x occurring only globally.

- $k \vDash \forall xA$ iff $k \vDash \top \to A[\overline{d}/x]$ for all $d \in D$. (Global quantification)

➢ Otherwise…

- $k \vDash \forall xA$ iff $k \vDash E(\overline{d}) \to A[\overline{d}/x]$ for all $d \in D$. (Local quantification)

➢ Then, $\vDash$ persists.

Theorems

Validities

- A is *valid in* W ($\models^{V}_{W} A$) $\Leftrightarrow$ forced at every node in W.
- $\mathcal{W}$: the class of all models of $\mathbf{SFQ}$.
- A is *valid in* $\mathcal{W}$ ($\models^{V}_{\mathcal{W}} A$) $\Leftrightarrow$ valid in all $W \in \mathcal{W}$.
  - Valid in $\mathcal{W}$ are, e.g.,
    - $\neg A \vee \neg\neg A$ (Weak LEM),
    - $\sim\sim (A \vee \sim A)$ (Double-negated int-LEM),
    - $\sim\sim A \rightarrow A$ (int-DNE),
    - $\big((A \rightarrow B) \rightarrow A\big) \rightarrow A$ (Peirce's law) and
    - $\forall x E(x)$ (Whatever will "E"xist will "E"xist).

Theorems

Invalidities

- $A \vee \neg A$ (LEM) fails.
- $\neg\neg A \rightarrow A$ (DNE) fails.
- $\vDash^{V}_{\mathcal{W}} A \rightarrow B$ & $\vDash^{V}_{\mathcal{W}} A$ don't imply $\vDash^{V}_{\mathcal{W}} B$ (MP), but $\vDash^{V}_{\mathcal{W}} \sim\sim B$.

for any $k \in K$, $k \vDash \sim\sim A \Longrightarrow k \vDash A$.

  - ➢ A is *stable* in $W \in \mathcal{W}$ if $\sim\sim A \vDash^{V}_{W} A$.

Class ST

$N \in \mathrm{GN}$

$S ::= \top | N | S \wedge S | S \vee N | N \vee S | \mathrm{Form} \rightarrow \mathrm{Form} | \forall x \mathrm{Form}$

  - ➢ Every ST formula is stable in every $W \in \mathcal{W}$.
    - ➢ If $\vDash^{V}_{\mathcal{W}} A \rightarrow S$ & $\vDash^{V}_{\mathcal{W}} A$, then $\vDash^{V}_{\mathcal{W}} S$ (Stable MP).

Notes

Why Is **SFQ** Good?

(1) Conforms to Wright's philosophical principles.

- E.g., the "practical weak decidability": $\vDash_{W}^{V} \neg A \vee \neg\neg A$.

(2) Can formalise the conditions of "Wang's paradox".

- There is a model where

  1. $P(\underline{0})$,
  2. $\forall x\,(P(x) \to P(S(x)))$, and
  3. $\exists x \neg P(x)$

  are all valid.

Allowing induction: $\forall x P(x)$.

2. A Complete Proof System

Points of Natural Deduction **NFQ**

- ($\rightarrow$) & ($\forall$): involve $\sim\sim$.

$$\frac{\begin{array}{c}[A]\\ \vdots \\ \sim\sim B\end{array}}{A \rightarrow B}\rightarrow\text{I} \qquad \frac{A \rightarrow B \qquad A}{\sim\sim B}\rightarrow\text{E}$$

$$\frac{\sim\sim A[y/x]}{\forall x A}\forall\text{-glo I} \qquad \frac{\forall x A}{\sim\sim A[t/x]}\forall\text{-glo E}$$

- Local ($\forall$) & ($\exists$): involve E.

$$\frac{\begin{array}{c}[E(x)]\\ \vdots \\ \sim\sim A[y/x]\end{array}}{\forall x A}\forall\text{-loc I} \qquad \frac{\forall x A \qquad E(t)}{\sim\sim A[t/x]}\forall\text{-loc E}$$

- $\sim$ under GN is $\neg$.

$$\frac{\begin{array}{c}\text{GN}\\ \vdots \\ \sim A\end{array}}{\neg A}\neg\text{I}_2$$

Natural Deduction **NFQ** in Full

- (∧) & (∨): classical.
- (⊤) & (⊥): usual. $\dfrac{}{\top}\ \top\mathrm{I}$ $\qquad$ $\dfrac{\bot}{A}\ \bot\mathrm{E}$
- (Strictness): $\dfrac{P(c)}{E(c)}\ \mathrm{STR}_1$ $\qquad$ $\dfrac{E(f(c))}{E(c)}\ \mathrm{STR}_2$
- (Stability): $\dfrac{\sim\sim S}{S}\ \mathrm{ST}$

Systems

Natural Deduction $\mathbf{NFQ}$ in Full

$$\dfrac{\begin{matrix}[A]\\ \vdots\\ {\sim\sim} B\end{matrix}}{A \to B}\ \to\mathrm{I} \qquad \dfrac{A \to B \qquad A}{{\sim\sim} B}\ \to\mathrm{E}$$

$$\dfrac{\sim N}{\neg N}\ \neg\mathrm{I}_1 \qquad \dfrac{\begin{matrix}\color{red}{\mathrm{GN}}\\ \vdots\\ \sim A\end{matrix}}{\neg A}\ \neg\mathrm{I}_2$$

$$\dfrac{\sim A \qquad A}{\bot}\ \to\bot\mathrm{E} \qquad \dfrac{\neg A \qquad A}{\bot}\ \neg\mathrm{E}$$

$$\dfrac{{\sim\sim} A[y/x]}{\forall x A}\ \forall\text{-glo I} \qquad \dfrac{\begin{matrix}[E(x)]\\ \vdots\\ {\sim\sim} A[y/x]\end{matrix}}{\forall x A}\ \forall\text{-loc I}$$

$$\dfrac{\forall x A}{{\sim\sim} A[t/x]}\ \forall\text{-glo E} \qquad \dfrac{\forall x A \qquad E(t)}{{\sim\sim} A[t/x]}\ \forall\text{-loc E}$$

$$\dfrac{A[t/x]}{\exists x A}\ \exists\text{-glo I} \qquad \dfrac{A[t/x] \qquad E(t)}{\exists x A}\ \exists\text{-loc I}$$

$$\dfrac{\exists x A \qquad \begin{matrix}[A[y/x]]\\ \vdots\\ C\end{matrix}}{C}\ \exists\text{-glo E} \qquad \dfrac{\exists x A \qquad \begin{matrix}[A[y/x]]\ [E(x)]\\ \vdots\\ C\end{matrix}}{C}\ \exists\text{-loc E}$$

…with the variable conditions.

The **IQC** Strategy + Modifications

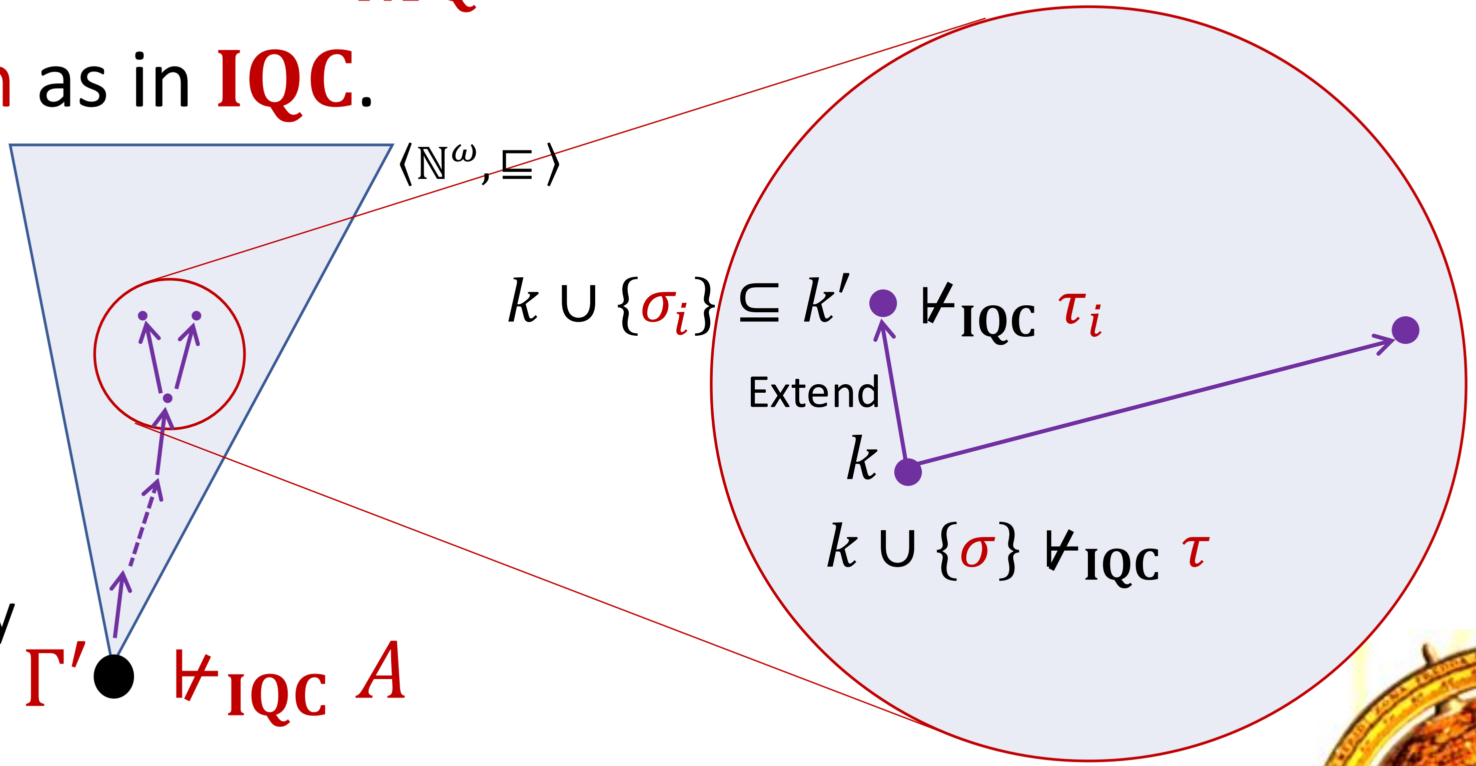
for any $W \in \mathcal{W}, k \in K, k \vDash \bigwedge\Gamma \Longrightarrow k \vDash A.$

- We can prove $\Gamma \vDash^{V}_{\mathcal{W}} A \Longrightarrow \Gamma \vdash_{\mathbf{NFQ}} A$ via the Henkin method
  - by contraposition as in **IQC**.

(i) Construct a theory-tree.

(ii) Prove the truth lemma:

- $B \in k \Longleftrightarrow k \vDash B$.



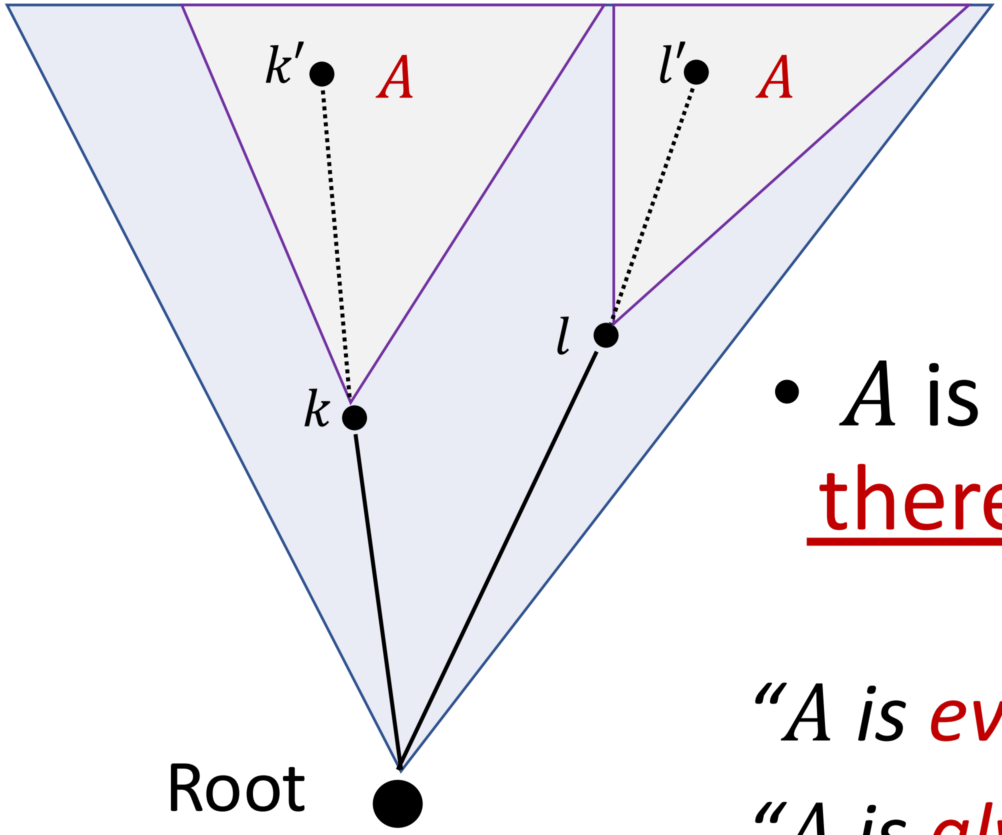
$\Gamma \nvdash_{\mathbf{IQC}} A$ — Extend to a prime theory → $\Gamma' \nvdash_{\mathbf{IQC}} A$

- ($\rightarrow$) and ($\forall$-loc) are taken care of.

3. $\mathbf{SFQ_P}$, the Logic with Prevalence

Ideas

Prevalence: Strong Verifiability



"$\vDash^{P}_{W} A$"

- A is *prevalent* if for any node k, there is a $k' \geq k$ such that $k' \vDash A$.

"A is eventually verified in any case."

"A is always open to verification."

Ideas

Prevalent Models and $\mathbf{SFQ_P}$

- A $W \in \mathcal{W}$ is *prevalent* if

    Satisfiable, i.e. "verifiable".

    1. for every formula A, $\vDash^{V}_{W} \neg\neg A$ implies $\vDash^{P}_{W} A$, and
    2. for all $d \in D$, $\vDash^{P}_{W} E(\overline{d})$.

- The communal interpretation.

    "A prevalent model stands for a community of individuals who report and share their findings."

➢ $\mathcal{W}_{\mathrm{P}}$: the class of the prevalent models.

➢ The logic $\mathbf{SFQ_P}$.

Theorems

Prevalence in $\mathcal{W}_{\mathrm{P}}$ Is Classical

- Classically calculable: in any $W \in \mathcal{W}_{\mathrm{P}}$,
  - $\vDash^P_W A \wedge B$ iff $\vDash^P_W A$ and $\vDash^P_W B$.
  - $\vDash^P_W A \vee B$ iff $\vDash^P_W A$ or $\vDash^P_W B$.
  - $\vDash^P_W A \rightarrow B$ iff $\nvDash^P_W A$ or $\vDash^P_W B$.
  - $\vDash^P_W \neg A$ iff $\nvDash^P_W A$ iff $\vDash^P_W \sim A$.
  - $\vDash^P_W \forall x A$ iff $\vDash^P_W A(d)$ for all $d \in \mathrm{D}$.
  - $\vDash^P_W \exists x A$ iff $\vDash^P_W A(d)$ for some $d \in D$.

"Truths in a candy bin".

- Unique future.
- Ignores the verification-order.

Model Contraction

- $\mathcal{W}_{\mathrm{P2}} \coloneqq$ 2-node prevalent models.
  - ➢ $\Gamma \vDash^{V}_{\mathcal{W}_{\mathrm{P}}} A \Leftrightarrow \Gamma \vDash^{V}_{\mathcal{W}_{\mathrm{P2}}} A$.
- $\mathcal{W}_{\mathrm{P2}\emptyset} \coloneqq$ 2-node prevalent preconstructive models.
  - ➢ $\vDash^{V}_{\mathcal{W}_{\mathrm{P}}} A \Leftrightarrow \vDash^{V}_{\mathcal{W}_{\mathrm{P2}\emptyset}} A$.

- $W = \langle K, \leq, D, J, v \rangle$ is *preconstructive* if $r \nvDash E(c)$ for any closed $c \in \mathcal{L}(D)$.

4. $\mathbf{SFQ_P}$ without $\neg$: Comparison with Other Logics

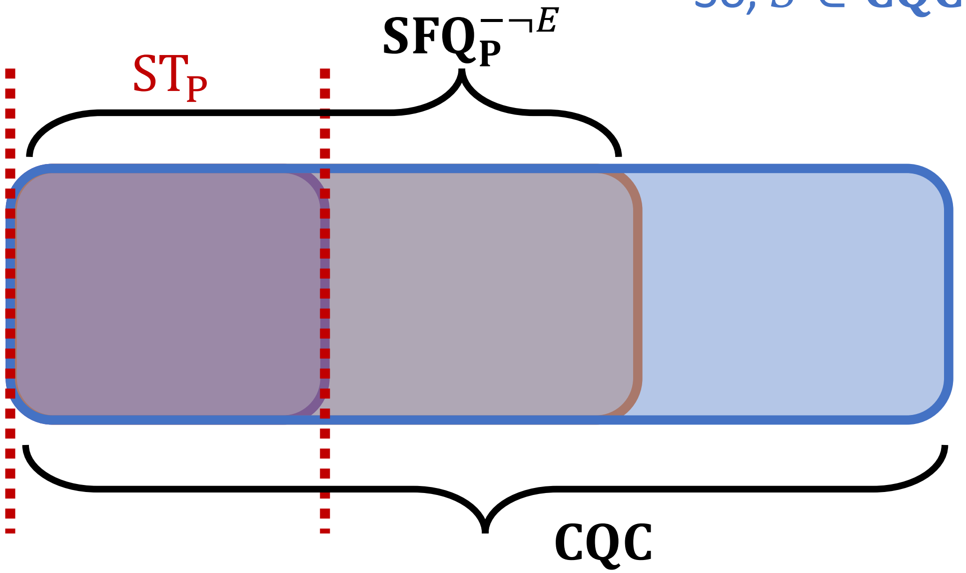
(No global quantification considered.)

Theorems

Comparison with $\mathbf{CQC}$

- $A \in \mathbf{CQC} \Leftrightarrow \sim\sim A \in \mathbf{SFQ}_{\mathbf{P}}^{--\neg E}$.
- If $S \in \mathrm{ST}_{\mathrm{P}}$, then $S \in \mathbf{CQC}$ implies $S \in \mathbf{SFQ}_{\mathbf{P}}^{--\neg E}$.

So, $S \in \mathbf{CQC} \Leftrightarrow S \in \mathbf{SFQ}_{\mathbf{P}}^{--\neg E}$.



$ST_P ::= S | S' \vee S' | \exists x S'$,
where $S \in ST$ & $\exists x S'$ is global.

- $A \in \mathbf{CQC} + \forall x E \Leftrightarrow \sim\sim A \in \mathbf{SFQ}_{\mathbf{P}}^{--\neg}$.
- If $S \in \mathrm{ST}_{\mathrm{P}}$, then $S \in \mathbf{CQC} + \forall x E$ implies $S \in \mathbf{SFQ}_{\mathbf{P}}^{--\neg}$.

So, $S \in \mathbf{CQC} + \forall x E \Leftrightarrow S \in \mathbf{SFQ}_{\mathbf{P}}^{--\neg}$.

Theorems

Comparison with **HTCD**

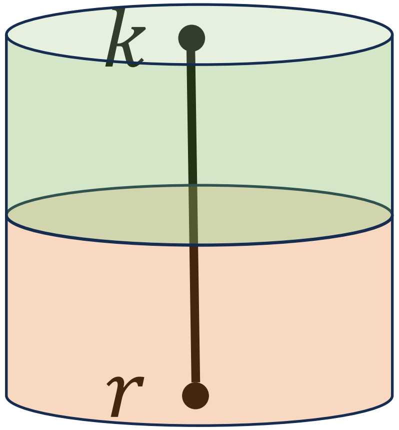
Universal-closed

- **HTCD** $=$ **IQC** $+ \underline{A \vee (A \to \mathrm{B}) \vee \sim \mathrm{B}}$ (HT, Here-and-there)

  x not free in C

  $+ \underline{\forall x(C \vee A) \to C \vee \forall x A}$ (CD, Constant Domain).
- Is sound & complete w.r.t. the class of the intuitionistic 2-node models with constant domains.
- $*$ on $\mathrm{Form}[\mathcal{L}]^{-\neg E}$: $P^* \coloneqq P$, $(A \blacksquare B)^* \coloneqq A^* \blacksquare B^*$, $\blacksquare \in \{\wedge, \vee, \to\}$.

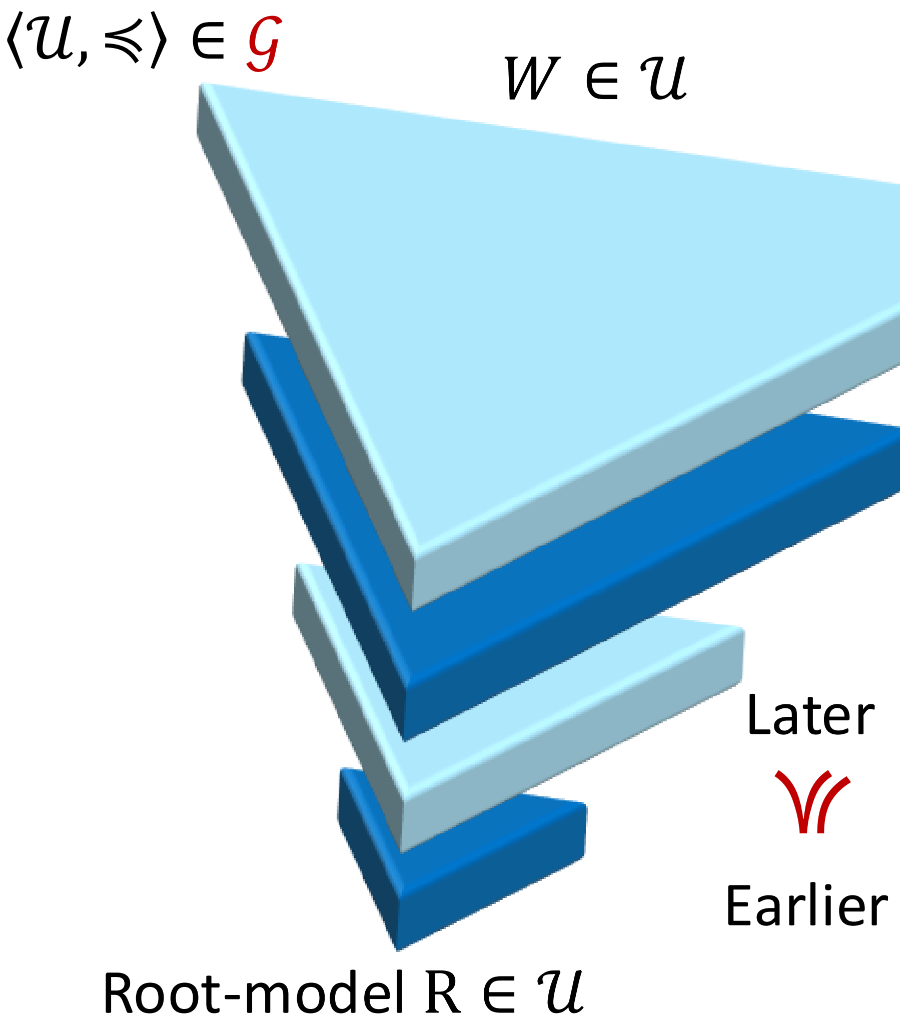
  $(\forall x A)^* \coloneqq \forall x(A^*)$, $(\exists x A)^* \coloneqq \exists x(\sim\sim A^*)$.
- $[\mathbf{HTCD}]^* \subseteq \mathbf{SFQ}_{\mathbf{P}}^{-\neg E}$.
  - $((A \to B) \to A) \to A \notin$ **HTCD**.
  - $\exists x(Px \to Px) \notin \mathbf{SFQ}_{\mathbf{P}}^{-\neg E}$.



5. $\mathbf{SFQ_P}$ without $\neg$: Correspondence with $\mathbf{IQC}$

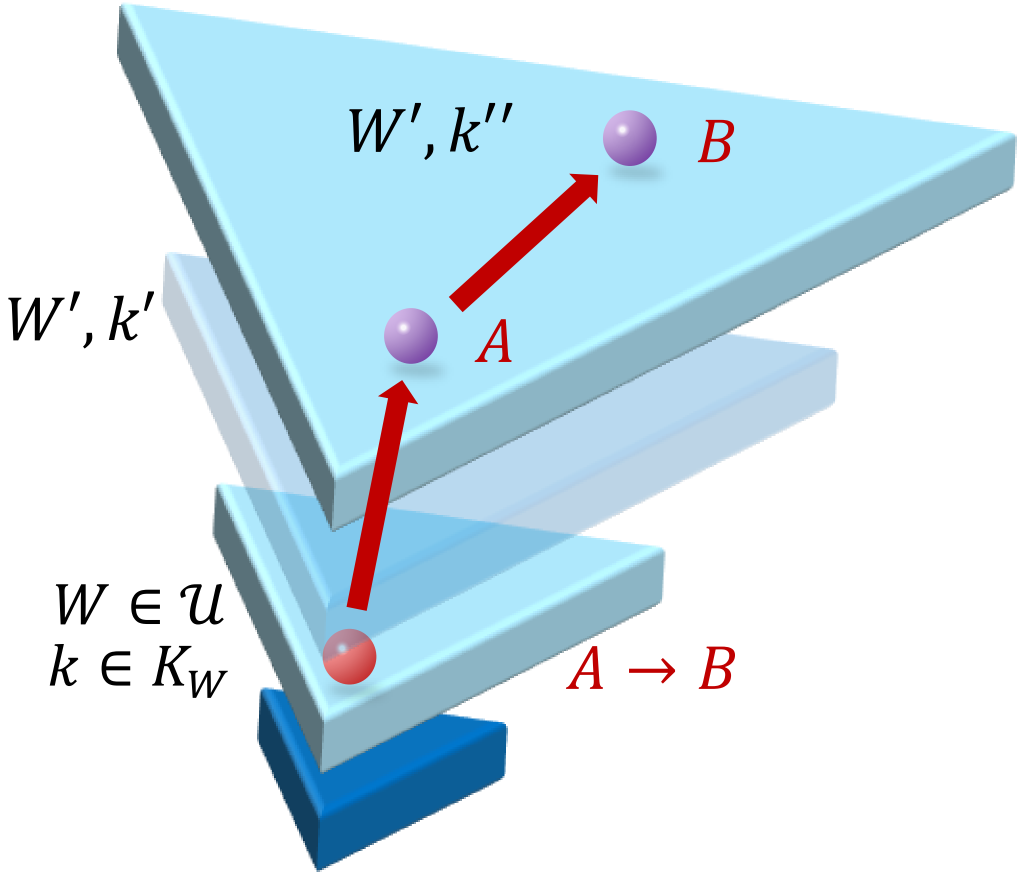
(No global quantification considered.)

Generation Structures



- A generation structure $\langle \mathcal{U}, \preccurlyeq \rangle \in \mathcal{G}$. "g-structure".
  - $\mathcal{U} \subseteq \mathcal{W}_{\mathrm{P,fin}}$.
  - $\preccurlyeq$: the initial part relation on $\mathcal{U}$.
- Let $\langle \mathcal{U}, \preccurlyeq \rangle$ be an intuitionistic frame.
  - Stands for the same community from various generations with increasing power.

The Generation Semantics



- Define $W, k \Vdash_G A$ "naturally":

(→) $W, k \Vdash_G A \to B \Leftrightarrow$ for any $W' \succcurlyeq W$ and $k' (\geq k)$ of W', if $W', k' \Vdash_G A$, then for some $k'' (\geq k')$ of W', $W', k'' \Vdash_G B$.

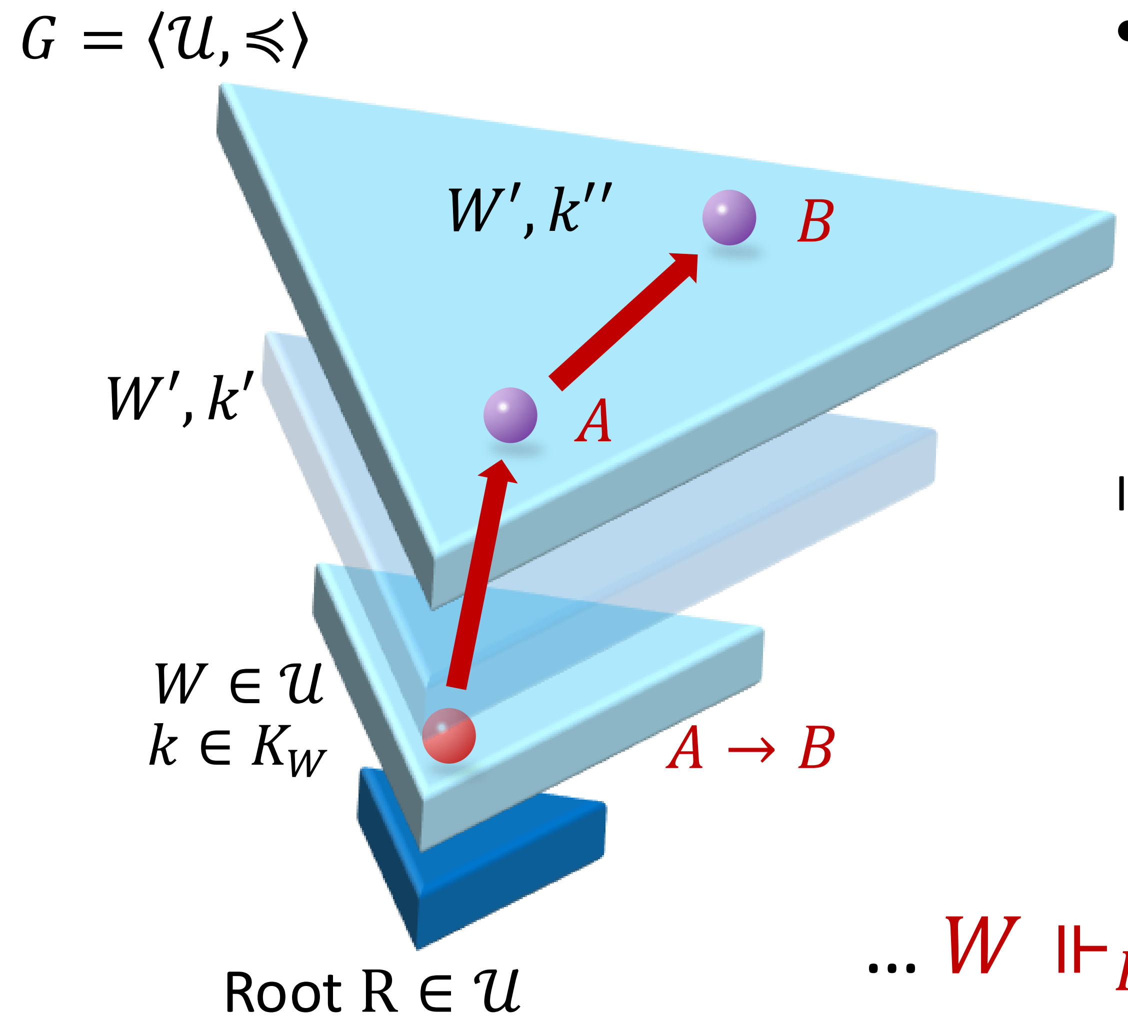
(∀) $W, k \Vdash_G \forall x\, A \Leftrightarrow$ for any $W' \succcurlyeq W$ and d in the domain of W', $W', k \Vdash_G E(\bar{d}) \to A[\bar{d}/x]$.

(∃) $W, k \Vdash_G \exists x A \Leftrightarrow W, k \Vdash_G E(\bar{d}) \wedge A[\bar{d}/x]$ for some d in the domain of W.

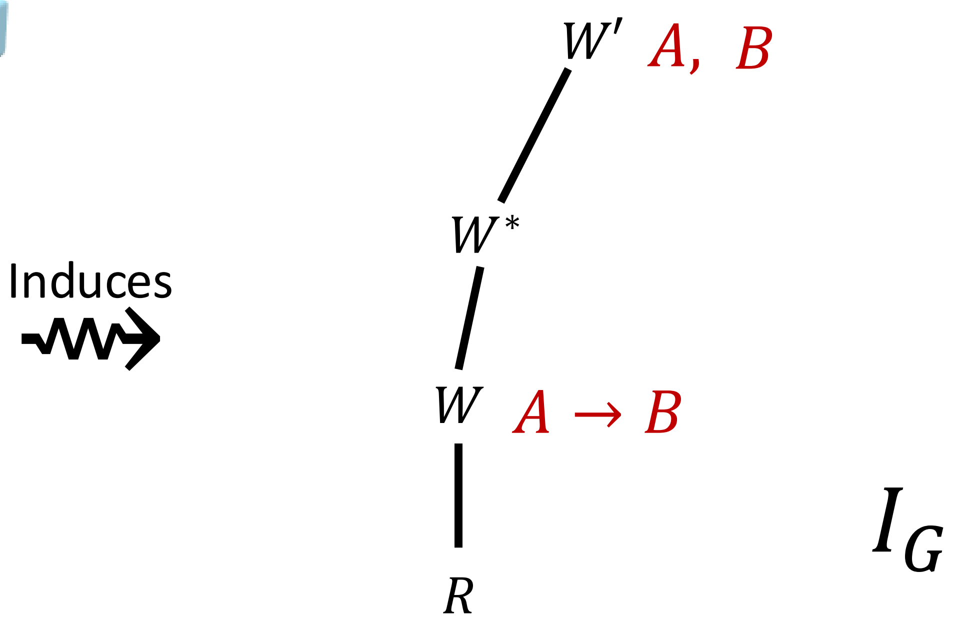
➢Then, $\Vdash$ persists both in $\preccurlyeq$ and $\leq$.

➢$\Vdash$: SF-like forcing with increasing power.

$\mathbf{SFQ_P^{\neg E}}$-IQC Correspondence (i)



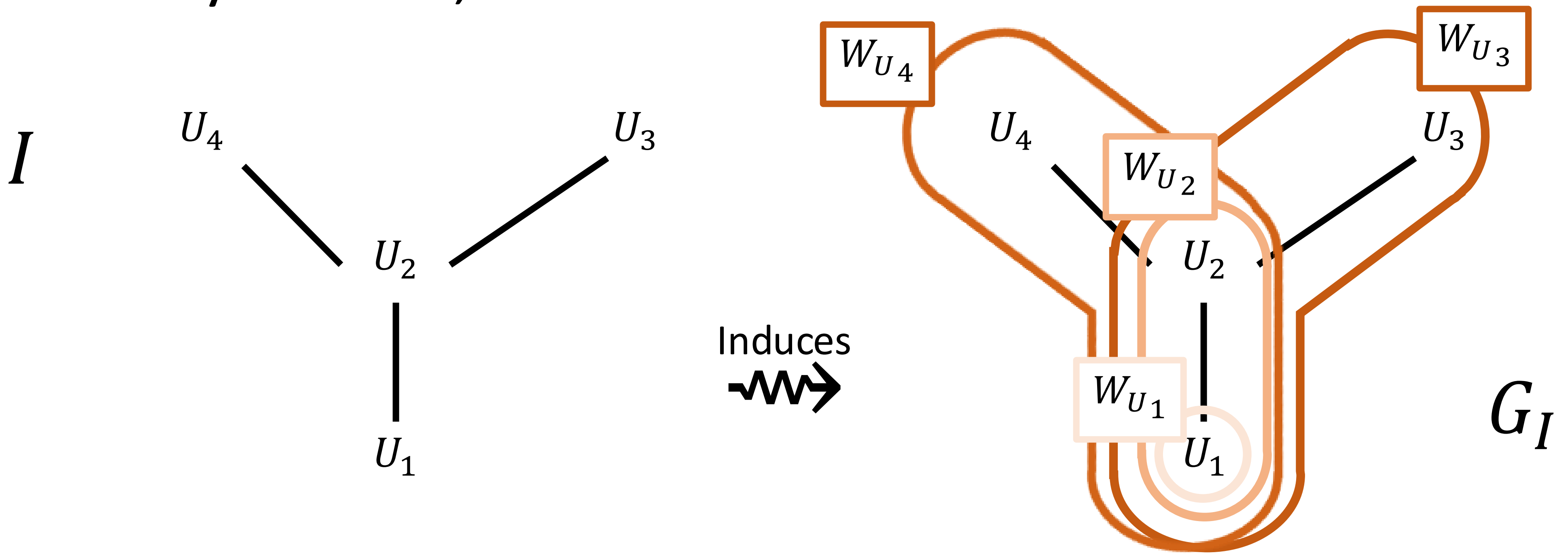
- Every $G = \langle \mathcal{U}, \preccurlyeq \rangle$ induces an **IQC** model I_G s.t., for any $W \in \mathcal{U}$, …



… $W \Vdash_{I_G} A$ iff $W, k \Vvdash_G A$ for some k of W.

$\mathbf{SFQ}_{\mathbf{P}}^{\neg\neg E}$-$\mathbf{IQC}$ Correspondence (ii)

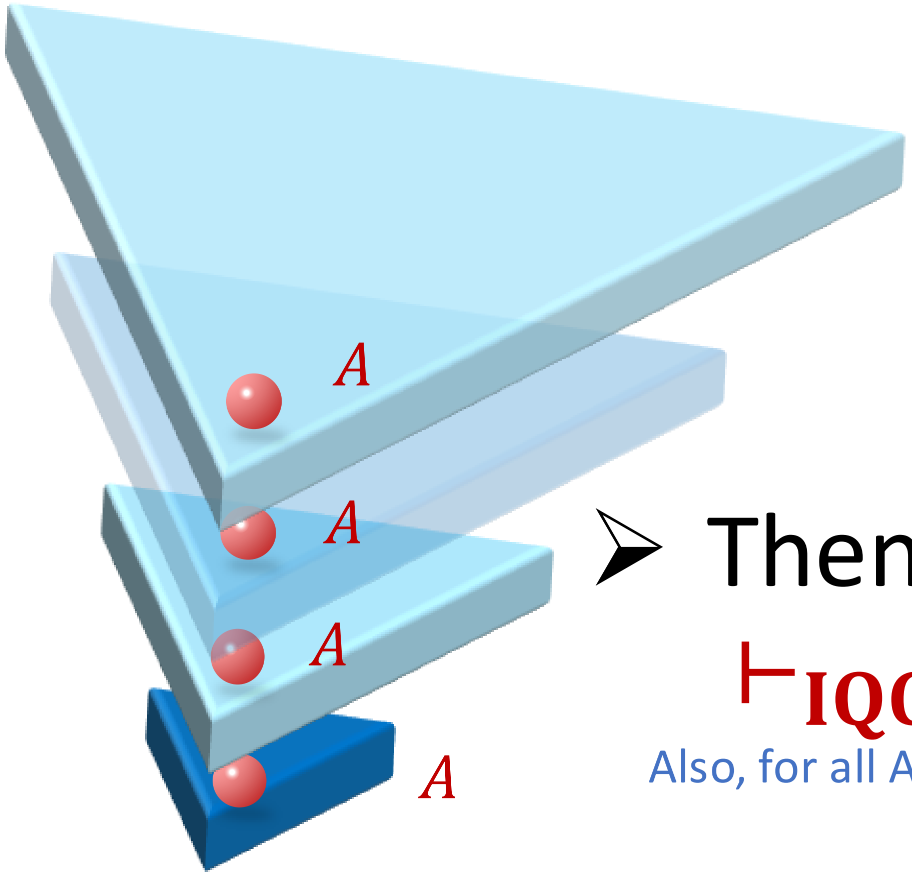
- Any **IQC** model $I = \langle U^*, \preccurlyeq^*, v^* \rangle$ induces a g-structure G_I s.t., for any $U \in U^*$, …



… $U \Vdash_I A$ iff $W_U, U \Vvdash_{G_I} A$.

$\mathcal{G}_C^{-E\neg} = \mathbf{IQC}$

& $\mathcal{G}_C^{-\neg} = \mathbf{IQC} + \forall x E$



- A is *valid in G* $\in \mathcal{G}$ if $W, k \Vdash_G A$ for every pair (W, k).
- $W \in \mathcal{W}_{\mathrm{P,fin}}$ is *postconstructive* if $\vDash_W^V E(t)$ for all closed terms t.

In the default language.

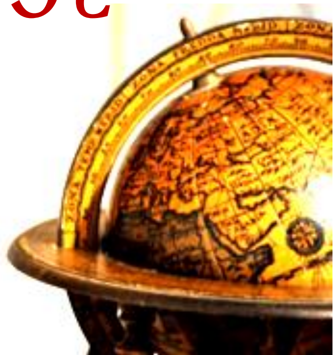
➢ Class $\mathcal{G}_C$.

➢ Then, for all A without E and $\neg$, $\vdash_{\mathbf{IQC}} A$ iff A is valid in all $G \in \mathcal{G}_C$.

Also, for all A without $\neg$, $\forall x E \vdash_{\mathbf{IQC}} A$ iff A is valid in all $G \in \mathcal{G}_C$.

The modality bridging principle, formalised

➢"In principle = in practice + extension in power".



Summary

- **SFQ**: Reconstruction of Crispin Wright's *strict finitistic logic*.
  - A predicate logic with the existence predicate & 2 modes of quantification.
  - Consistently formalises "Wang's Paradox".
- **SFQ$_P$**: The logic under prevalence.
  - Comparable with intermediate logics.
  - Formalises the modality bridging principle.