How can we add classical negation to intuitionistic sequent calculus?

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- This work is a joint work with Katsuhiko Sano (Hokkaido University)
- The contents of this talk were already published:

➢ Toyooka, M. & Sano, K.: Combining intuitionistic and classical propositional logic: Gentzenization and Craig interpolation. *Studia Logica*, 112:1091—1121, 2024.

➢ Toyooka, M. & Sano, K.: Combining first-order classical and intuitionistic logic. *EPTCS*, 358:25—40, 2022.

➢ Toyooka, M. & Sano, K.: How can we avoid Popper's collapsing problem and have Craig interpolation? Annals of the Japan Association for Philosophy of Science, 33:145—162, 2024.

Our Contribution

- proposes a method to add classical negation to intuitionistic sequent calculus.
- reveals ``the core” of the right rule for intuitionistic implication.

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Motivation

- Debates between intuitionists and classists
  - ✓Philosophy of Mathematics
  - ✓Philosophy of Logic
  - ✓Philosophy of Language

- Two ways to formulate the debates:
  1. Disagreeing about valid logical laws.
  2. Attaching different meanings to logical connectives.
- The second view was come up with by Quine (1986).
- This talk is based on the second view.

- Trying to provide a system containing both intuitionistic and classical connectives is a natural direction of the research.
- This talk focuses on the concept of negation.
  - ✓ Classical negation: $\neg_c$
  - ✓ Intuitionistic negation: $\neg_i$
- What is required: conservativeness
  - ✓ $\neg_c$($\neg_i$) behaves classically (intuitionistically).

$A \lor \neg_c A$: derivable
$A \lor \neg_i A$: underivable

- Maehara’s mLJp: obtained from LKp by restricting the right rule for implication to the following form:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,.$$

- Right rule for negation in mLJp:

$$\frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A}\,.$$

- Straightforward idea: Adding $\neg_c$ to mLJp.
- The left and right rules for $\neg_c$:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_c A} \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg_c A, \Gamma \Rightarrow \Delta} .$$

- However, this idea does not work well.

$$\dfrac{\dfrac{A \Rightarrow A}{A, \neg_{\mathtt{i}} A \Rightarrow}}{\neg_{\mathtt{i}} A \Rightarrow \neg_{\mathtt{c}} A} \qquad \dfrac{\dfrac{A \Rightarrow A}{A, \neg_{\mathtt{c}} A \Rightarrow}}{\neg_{\mathtt{c}} A \Rightarrow \neg_{\mathtt{i}} A}$$

- The system is not conservative over IPC.

- This problem is called the collapsing problem (Popper 1948, Williamson 1988, Gabbay 1996).
- The problem is regarded as a difficulty to ``codify'' the debates between the two camps.
- Williamson argued that this problem revealed that Quine's view was wrong.

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- Various methods to avoid the collapsing problem were already provided.

✓Many of the methods: semantic

✓Few of the methods: proof-theoretic

- Most of the proof-theoretic methods employs expanded notions of a sequent.

✓Structured sequent (Lucio 2000)

✓Hypersequent (Kurokawa 2009)

✓Sequent with ``mode” (Liang & Miller 2013)

- This section provides a sequent calculus for a logic with the two negations, where the ordinary notion of a sequent is employed.
- Our calculus is sound and complete to the Kripke semantics proposed by Humberstone (1979) and del Cerro and Herzig (1996).

Syntax

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to_{\mathtt{i}} A \mid \neg_{\mathtt{c}} A,$$

$$\text{where } p \in \mathsf{Prop}$$

- $\neg_{\mathrm{i}} A$, $A \to_{\mathrm{c}} B$, and T are abbreviations of $A \to_{\mathrm{i}} \bot$, $\neg_{\mathrm{c}} A \vee B$, and $\bot \to_{\mathrm{i}} \bot$, respectively.

- Intuitionistic syntax: Prop, $\bot$, $\wedge$, $\vee$, $\to_{\mathrm{i}}$.
- Classical syntax: Prop, $\bot$, $\wedge$, $\vee$, $\neg_{\mathrm{c}}$.

Kripke Semantics for C+J
by (Humberstone 1979).

A model is a tuple $M = (W, \leq, V)$ where:

- $\leq$ is a partial order,
- V is a valuation satisfying persistency:

$$w \in V(p) \text{ and } w \leq v \text{ imply } v \in V(p).$$

- $M, w \vDash A \rightarrow_{\mathrm{i}} B \Leftrightarrow$
  For all v ($w \leq v$ & $M, v \vDash A$ imply $M, v \vDash B$).
- $M, w \vDash \neg_{\mathrm{c}} A \Leftrightarrow M, w \nvDash A$.

Definition

A formula A is valid in a Kripke model $M = (W, \leq, V)$ if $M, w \vDash A$ for all $w \in W$.

A is valid in all Kripke models iff $\Rightarrow A$ is valid.

Definition

A sequent $\Gamma \Rightarrow \Delta$ is valid if for any Kripke model $M = (W, \leq, V)$ and any $w \in W$, whenever $M, w \vDash C$ for all $C \in \Gamma$ then $M, w \vDash D$ for some $D \in \Delta$.

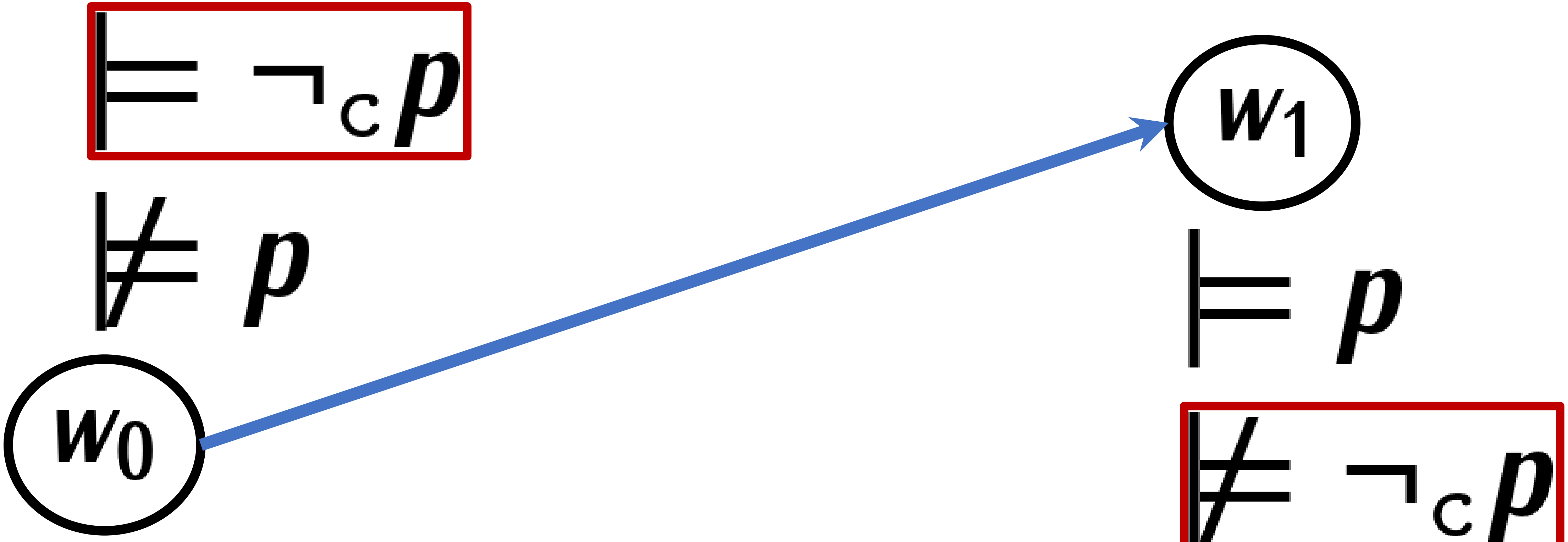
Definition

A formula A is persistent if it satisfies the following for any Kripke model $M = (W, \leq, V)$ and $w, v \in W$:

$$M, w \vDash A \text{ and } w \leq v \text{ imply } M, v \vDash A.$$

Proposition

The formula $\neg_c p$ is not persistent.



$\models \neg_c p$
$\not\models p$
w_0
w_1
$\models p$
$\not\models \neg_c p$

- The rule

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to_{\mathtt{i}} B}$$

does not preserve the validity of a sequent.

- For example, $\mathrm{T}, \neg_{\mathrm{c}} p \Rightarrow \neg_{\mathrm{c}} p$ is valid, but $\neg_{\mathrm{c}} p \Rightarrow \mathrm{T} \to_{\mathrm{i}} \neg_{\mathrm{c}} p$ is not.
- Thus, we need to restrict this rule.
- How should we restrict?

$$\frac{A, C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow A \rightarrow_{\mathtt{i}} B}$$

$$\frac{A, C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m \quad \Rightarrow B}{C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m \quad \Rightarrow A \to_{\mathtt{i}} B}$$

Right rule for implication in a sequent calculus for the logic of strict implication S4 (Kripke 1959), (Okada 1988), (Kashima 1999).

$$\frac{A, C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m \qquad \Rightarrow B}{C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m \qquad \Rightarrow A \rightarrow_{\mathtt{i}} B}$$

Right rule for implication in a sequent calculus for the logic of strict implication S4 (Kripke 1959), (Okada 1988), (Kashima 1999).

- Intuitionistic Kripke semantics w/o persistency of valuation,
- Any formula of the form $C \rightarrow D$ is persistent.

$$\frac{A, C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow A \rightarrow_{\mathtt{i}} B}$$

Reflecting persistency of a propositional variable.

- mLJp′: the calculus obtained from mLJp by replacing the right rule for implication with the restricted one:

Theorem New!

mLJp′ has the same derivability as mLJp.

- Original rule is admissible in mLJp′.
- The restricted rule reveals the ``core” of the original right rule for implication.

- G(C+J): mLJp′+

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_{\mathtt{c}} A} \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg_{\mathtt{c}} A, \Gamma \Rightarrow \Delta}\ .$$

- Rules for $\neg_{\mathrm{i}}$ in G(C+J):

$$\frac{A, C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow}{C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow \neg_{\mathtt{i}} A}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg_{\mathtt{i}} A, \Gamma \Rightarrow \Delta}$$

$$\dfrac{\dfrac{A \Rightarrow A}{A, \neg_{\mathtt{i}} A \Rightarrow}}{\neg_{\mathtt{i}} A \Rightarrow \neg_{\mathtt{c}} A} \qquad \dfrac{\dfrac{A \Rightarrow A}{A, \neg_{\mathtt{c}} A \Rightarrow}}{\neg_{\mathtt{c}} A \Rightarrow \neg_{\mathtt{i}} A}$$

$$\dfrac{\neg_{\mathtt{c}} p, \top \Rightarrow \neg_{\mathtt{c}} p}{\neg_{\mathtt{c}} p \Rightarrow \top \rightarrow_{\mathtt{i}} \neg_{\mathtt{c}} p}$$

Soundness New!

If $\vdash \Gamma \Rightarrow \Delta$, then $\vDash \Gamma \Rightarrow \Delta$.

By induction on the construction of a derivation.

Completeness New!

If $\vDash \Gamma \Rightarrow \Delta$, then there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash \Gamma' \Rightarrow \Delta'$.

By a canonical model argument provided by Humberstone (1979).

Cut Elimination New!

If $\vdash \Gamma \Rightarrow \Delta$, then $\vdash \Gamma \Rightarrow \Delta$ with no application of (Cut).

By Kashima's (2009) method using the extended cut rule:

$$\frac{\Gamma \Rightarrow \Delta, A^m \quad A^n, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma},$$

where A^l means l times repetition of A and $m, n \geq 0$.

We can avoid difficult cases occurring in the case of mLJp!

Corollary

G(C+J) enjoys the subformula property.

- Intuitionistic syntax: Prop, $\bot$, $\wedge$, $\vee$, $\rightarrow_{\mathrm{i}}$.
- Classical syntax: Prop, $\bot$, $\wedge$, $\vee$, $\neg_{\mathrm{c}}$.

Corollary

G(C+J) is a conservative extension of both mLJp and LKp.

Theorem New!

All the following items are equivalent:

1. $\vDash A$,
2. $\vdash \Rightarrow A$,
3. $\vdash \Rightarrow A$ w/o cut rule.

First-Order Expansion

- We expand the syntax by classical universal quantifier $\forall_{\mathrm{c}}$.

$$M, w \vDash \forall_{\mathrm{c}} x\, A \Leftrightarrow M, w \vDash A[\underline{d}/x] \text{ for all } d \in D(w).$$

(Lucio 2000)

- $\forall_{\mathrm{i}} x \mathrm{A}$ and $\exists x A$ can be defined as $\top \to_{\mathrm{i}} \forall_{\mathrm{c}} x A$ and $\neg_{\mathrm{c}} \forall_{\mathrm{c}} x \neg_{\mathrm{c}} A$, respectively.

- G(FOC+J): G(C+J)+

$$\frac{\Gamma \Rightarrow \Delta, A[z/x]}{\Gamma \Rightarrow \Delta, \forall_{\mathsf{c}} xA} \ (\Rightarrow \forall_{\mathsf{c}})^{\dagger} \qquad \frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall_{\mathsf{c}} xA, \Gamma \Rightarrow \Delta} \ (\forall_{\mathsf{c}} \Rightarrow)$$

†: z does not occur free in the lower sequent.

- G(FOC+J) is sound and strongly complete.
- Cut elimination holds for G(FOC+J) .

Summary of Our Method

- The recipe to obtain G(C+J):

1. restrict the right rule for implication to its core.
2. add rules for classical negation.

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- The recipe to obtain G(C+J):

1. restrict the right rule for implication to its core.
2. add rules for classical negation.

- Seeing IPC as

strict implication S4 + persistency of a valuation.

- Another way to see G(C+J):

LKp + special implication $\rightarrow_{i}$ (cf. del Cerro & Herzig 1996).

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Applications of Cut Elimination

Also holds for G(FOC+J)

- Direct argument for the decidability.

✓via S4 (del Cerro and Herzig 1996)

Craig Interpolation New!

If $\vdash A \Rightarrow B$, then there is a formula C s.t.

- both $\vdash A \Rightarrow C$ and $\vdash C \Rightarrow B$ and
- Prop $(C) \subseteq$ Prop $(A) \cap$ Prop (B).

By Maehara's method similar to the one for LKp (not mLJp).

Comparison with Other Logics

- IPC+~ (De 2013, De & Omori 2014):

$$M, w \vDash {\sim}A \Leftrightarrow M, g \nvDash A.$$

- IPC+$\neg_\Omega$ (Humberstone 2006, Niki & Omori 2021):

$$M, w \vDash \neg_\Omega A \Leftrightarrow w = g \text{ implies } M, w \nvDash A.$$

- The logics are defined by the set of valid formulas.

- All the three logics are conservative over both IPC and CPC.
- All the three logics are incomparable.
- $\sim$ and $\neg_\Omega$ preserve all the schemata of theorems of IPC, but $\neg_c$ does not.

$$\not\models A \to (B \to A)$$

$$\models p \to (B \to p)$$

$$\models (A \to C) \to (B \to (A \to C))$$

Persistency for p

Transitivity

Generality of Our Approach

- The restriction method can be applied to an expansion of IPC without persistency.
- The addition of the operator M also breaks the persistency of IPC (Gabbay 1982):

$$\mathfrak{M}, w \models \mathrm{M}A \quad \text{iff} \quad \text{for some } v \in W : w \leqslant v \text{ and } \mathfrak{M}, v \models A.$$

- Wansing (1996 & 1999) and Omori (2016) added M to Nelson's logics.
- Sequent Calculus for a Nelson's logic + M (Sano & Toyooka, forthcoming):

$$\frac{\Lambda, C_1 \to D_1, \ldots, C_n \to D_n, A \Rightarrow B, \mathsf{M}\Delta}{\Lambda, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B, \mathsf{M}\Delta} \ (\Rightarrow\to)$$

$$\frac{\Lambda, C_1 \to D_1, \ldots, C_n \to D_n, A \Rightarrow \mathsf{M}\Delta}{\Lambda, C_1 \to D_1, \ldots, C_n \to D_n, \mathsf{M}A \Rightarrow \mathsf{M}\Delta} \ (\mathsf{M}\Rightarrow),$$

where Λ is a finite multiset of formulas of the form p or $\sim p$.

Our Contribution

- proposes a method to add classical negation to intuitionistic sequent calculus.
- reveals ``the core” of the right rule for intuitionistic implication.

Thank You!