Propositional Dynamic Logic has Craig Interpolation

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Propositional Dynamic Logic (PDL)

Syntax

$$\varphi ::= p \mid \bot \mid \neg \varphi \mid \varphi \wedge \varphi \mid [\alpha]\varphi$$

$$\alpha ::= a \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \varphi ?$$

where $p \in Var$ and $a \in At$

Models

$M = (W, \mathcal{R}, V)$ where

- W : set of states/worlds
- $\mathcal{R} = (R_a)_a$ is a family of binary relations on W
- $V : Prop \longrightarrow 2^W$

Propositional Dynamic Logic (PDL)

$\mathcal{M}, w \Vdash p$ iff $w \in V(p)$

$\mathcal{M}, w \Vdash \bot$ never

$\mathcal{M}, w \Vdash \neg \varphi$ iff $\mathcal{M}, w \nVdash \varphi$

$\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$

$\mathcal{M}, w \Vdash [\alpha] \varphi$ iff $\mathcal{M}, v \Vdash \varphi$ for all $(w, v) \in R_\alpha$

$R_{\alpha ; \beta} = R_\alpha ; R_\beta$ $\qquad$ $R_{\alpha^*} = (R_\alpha)^*$

$R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$ $\qquad$ $R_{\varphi ?} = \{(w, w) \in W^2 \mid w \Vdash \varphi\}$

Does PDL have CIP?

Is it the case that for every $\models \varphi \to \psi$ there exists a formula θ s.t.

- $\models \varphi \to \theta$ and $\models \theta \to \psi$
- $Voc(\theta) \subseteq Voc(\varphi) \cap Voc(\psi)$?

Does PDL have CIP?

Is it the case that for every $\models \varphi \to \psi$ there exists a formula θ s.t.

- $\varphi \models \theta$ and $\theta \models \psi$
- ~~$\models \varphi \to \theta$~~ and ~~$\models \theta \to \psi$~~

- $Voc(\theta) \subseteq Voc(\varphi) \cap Voc(\psi)$?

Does PDL have CIP?

Is it the case that for every $\models \varphi \to \psi$

there exists a formula θ s.t.

- $\varphi \models \theta$ and $\neg\psi \models \neg\theta$
- $\varphi \models \theta$ and $\cancel{\theta \models \psi}$
- $\cancel{\models \varphi \to \theta}$ and $\cancel{\models \theta \to \psi}$

- $Voc(\theta) \subseteq Voc(\varphi) \cap Voc(\psi)$?

Does PDL have CIP?

Is it the case that for every $\cancel{\models \varphi \to \psi}$ unsatisfiable $\varphi \wedge \neg \psi$

there exists a formula θ s.t.

- $\varphi \models \theta$ and $\neg\psi \models \neg\theta$
- $\varphi \models \theta$ and $\cancel{\theta \models \psi}$
- $\cancel{\models \varphi \to \theta}$ and $\cancel{\models \theta \to \psi}$

- $Voc(\theta) \subseteq Voc(\varphi) \cap Voc(\psi)$?

Does PDL have CIP?

unsatisfiable $\varphi \wedge \neg\psi$

Is it the case that for every ~~$\models \varphi \to \psi$~~

there exists a formula θ s.t.

- $\varphi \models \theta$ and $\neg\psi \models \neg\theta$
- $\varphi \models \theta$ and ~~$\theta \models \psi$~~
- ~~$\models \varphi \to \theta$~~ and ~~$\models \theta \to \psi$~~
- $Voc(\theta) \subseteq Voc(\varphi) \cap Voc(\neg\psi)$
- $Voc(\theta) \subseteq Voc(\varphi) \cap \cancel{Voc(\psi)}$?

Does PDL have CIP?

unsatisfiable $\varphi \wedge \psi$

Is it the case that for every $\cancel{\models \varphi \to \psi}$

there exists a formula θ s.t.

- $\varphi \models \theta$ and $\psi \models \neg\theta$
- $\varphi \models \theta$ and $\cancel{\theta \models \psi}$
- $\cancel{\models \varphi \to \theta}$ and $\cancel{\models \theta \to \psi}$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\varphi) \cap \cancel{\mathrm{Voc}(\psi)}$

?

Does PDL have CIP?

Is it the case that for every unsatisfiable $\varphi \wedge \psi$ there exists a formula θ s.t.

- $\varphi \models \theta$ and $\psi \models \neg\theta$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$?

Does PDL have CIP?

Is it the case that for every unsatisfiable $\Gamma_1 \cup \Gamma_2$ there exists a formula θ s.t.

- $\Gamma_1 \models \theta$ and $\Gamma_2 \models \neg\theta$
- $Voc(\theta) \subseteq Voc(\Gamma_1) \cap Voc(\Gamma_2)$?

Tableau Approved ✓

History: Does PDL have interpolation?

- Leivant (1981)

- Borzechowski (1988)

- Kowalski (2002)

[Lei:81], [Bor-88], [Kow02], [Kow04]

History: Does PDL have interpolation?

- Leivant (1981)

- Borzechowski (1988) ⟵ 3 Key ideas but... many problems

- Kowalski (2002)

The Approach

Maehara's method

on a

cyclic tableau system

with a

twist!

A tableau system for PDL

$$\frac{\varphi \wedge \psi, \Gamma}{\varphi, \psi, \Gamma} \ (\wedge)$$

$$\frac{\varphi \vee \psi, \Gamma}{\varphi, \Gamma \mid \psi, \Gamma} \ (\vee)$$

$$\begin{tikzcd}[column sep=small]
 & p \vee (q \wedge r), \neg p, \neg r \arrow[dl] \arrow[dr] \arrow[d, phantom, "(\vee)"] & \\
p, \neg p, \neg r & & q \wedge r, \neg p, \neg r \arrow[d, "(\wedge)"] \\
 & & q, r, \neg p, \neg r
\end{tikzcd}$$

Theorem: There exists a closed tableau for Γ iff
(sound & complete) Γ is unsatisfiable.

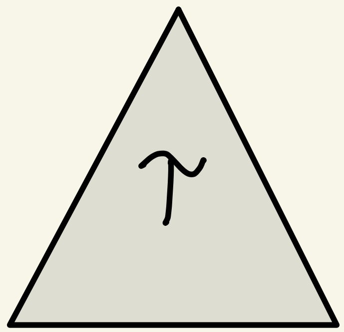
Towards Maehara

$\Gamma_1 \cup \Gamma_2$

unsatisfiable

$\rightsquigarrow$

$\Gamma_1 \cup \Gamma_2$



$\rightsquigarrow$ θ

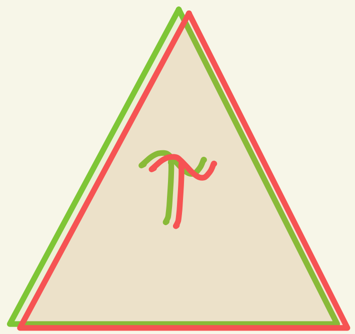
Towards Maehara

$\Gamma_1 \cup \Gamma_2$

unsatisfiable

$\rightsquigarrow$

$\Gamma_1 \, ; \, \Gamma_2$



$\rightsquigarrow \quad \theta$

A split tableau system for PDL

$\Gamma_1 ; \Gamma_2$ Split Sequents

$$\frac{\varphi \wedge \psi, \Gamma_1 ; \Gamma_2}{\varphi, \psi, \Gamma_1 ; \Gamma_2} (\wedge)_\ell \qquad \frac{\Gamma_1 ; \Gamma_2, \varphi \wedge \psi}{\Gamma_1 ; \Gamma_2, \varphi, \psi} (\wedge)_r$$

$$\frac{\varphi \vee \psi, \Gamma_1 ; \Gamma_2}{\varphi, \Gamma_1 ; \Gamma_2 \mid \psi, \Gamma_1 ; \Gamma_2} (\vee)_\ell \qquad \frac{\Gamma_1 ; \Gamma_2, \varphi \vee \psi}{\Gamma_1 ; \Gamma_2, \varphi \mid \Gamma_1 ; \Gamma_2, \psi} (\vee)_r$$

Theorem: There exists a closed tableau for $\Gamma_1 ; \Gamma_2$ iff
(sound & complete) $\Gamma_1 \cup \Gamma_2$ is unsatisfiable.

Maehara's Method

$$
\begin{tikzcd}
 & p \vee (q \wedge r) \,;\, \neg p, \neg r \arrow[dl] \arrow[dr, "(\vee)_\ell"] & \\
p \,;\, \neg p, \neg r & & q \wedge r \,;\, \neg p, \neg r \arrow[d, "(\wedge)_\ell"] \\
 & & q, r \,;\, \neg p, \neg r
\end{tikzcd}
$$

Maehara's Method

$$
\begin{tikzcd}[column sep=small]
& p \vee (q \wedge r) \,;\, \neg p, \neg r : p \vee r \arrow[dl, "(\vee)_\ell"] \arrow[dr] & \\
p \,;\, \neg p, \neg r : p & & q \wedge r \,;\, \neg p, \neg r : r \arrow[d, "(\wedge)_\ell"] \\
& & q, r \,;\, \neg p, \neg r : r
\end{tikzcd}
$$

Recall:

- $\Gamma_1 \models \theta$ and $\Gamma_2 \models \neg \theta$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\Gamma_1) \cap \mathrm{Voc}(\Gamma_2)$

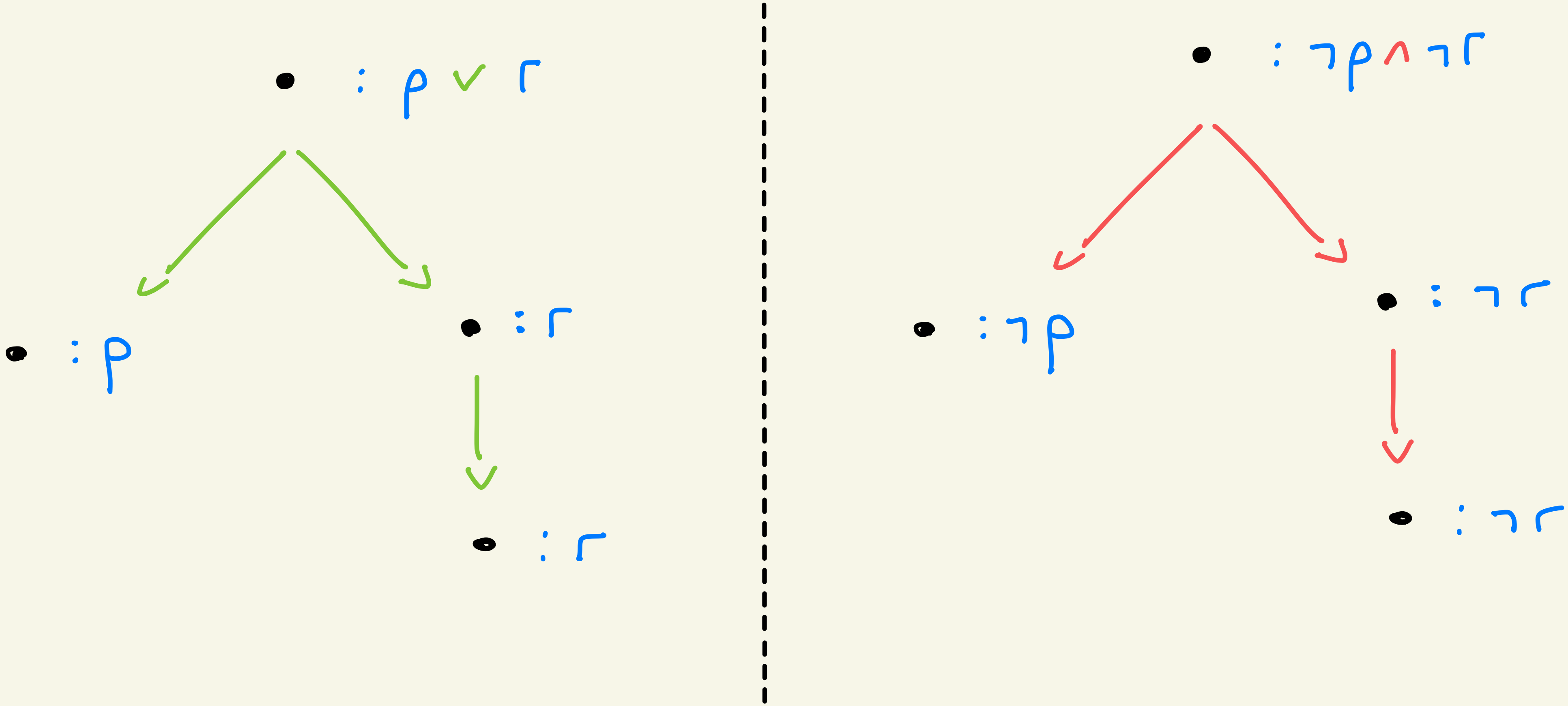
Maehara's Method

$$\begin{tikzcd}[column sep=small]
& \neg p, \neg r \,;\, p \vee (q \wedge r) : \neg p \wedge \neg r \arrow[dl] \arrow[dr] \arrow[d, phantom, "(\vee)_r"] & \\
\neg p, \neg r \,;\, p : \neg p & {} & \neg p, \neg r \,;\, q \wedge r : \neg r \arrow[d, "(\wedge)_r"] \\
& & \neg p, \neg r \,;\, q, r : \neg r
\end{tikzcd}$$

Recall:

- $\Pi_1 \models \theta$ and $\Pi_2 \models \neg\theta$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\Pi_1) \cap \mathrm{Voc}(\Pi_2)$

Maehara's Method



A split tableau system for PDL

$$\frac{\Gamma_1 ; \Gamma_2, \neg[a]\varphi}{(\Gamma_1)_a ; (\Gamma_2)_a, \neg\varphi}\ (M)_r \qquad \Delta_a = \{\varphi \mid [a]\varphi \in \Delta\}$$

$$\frac{\Gamma_1 ; \Gamma_2, \neg[\alpha^*]\varphi}{\Gamma_1 ; \Gamma_2, \neg\varphi \mid \Gamma_1 ; \Gamma_2, \neg[\alpha][\alpha^*]\varphi}\ (\neg *)_r$$

$$\frac{\Gamma_1 ; \Gamma_2, [\alpha^*]\varphi}{\Gamma_1 ; \Gamma_2, \varphi, [\alpha][\alpha^*]\varphi}\ (*)_r$$

Cyclic Maehara for PDL

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*] p$$

$$\Big\downarrow (M)_r$$

$$[a^*](p \wedge q) \,;\, \neg [a^*] p$$

$$\Big\downarrow (*)_l, (\wedge)_l$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a^*] p$$

$$\swarrow \quad (\neg *)_r \quad \searrow$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg p : p \qquad\qquad p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*] p : ?$$

Cyclic Maehara for PDL

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*]p \;:\; [a](p \wedge ?)$$

$$\big\downarrow (M)_r$$

$$[a^*](p \wedge q) \,;\, \neg [a^*]p \;:\; p \wedge ?$$

$$\big\downarrow (*)_l, (\wedge)_l$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a^*]p \;:\; p \wedge ?$$

$$\swarrow \quad (\neg *)_r \quad \searrow$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg p \;:\; p \qquad\qquad p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*]p \;:\; ?$$

Cyclic Maehara for PDL

$$p, q, [a][a^*](p \wedge q) ; \neg[a][a^*]p \;:\; [a](p \wedge ?) \equiv ?$$

$$\Big\downarrow (M)_r$$

$$[a^*](p \wedge q) ; \neg[a^*]p \;:\; p \wedge ?$$

$$\Big\downarrow (*)_l, (\wedge)_l$$

$$p, q, [a][a^*](p \wedge q) ; \neg[a^*]p \;:\; p \wedge ?$$

$$(\neg *)_r$$

$$p, q, [a][a^*](p \wedge q) ; \neg p \;:\; p \qquad\qquad p, q, [a][a^*](p \wedge q) ; \neg[a][a^*]p \;:\; ?$$

Cyclic Maehara for PDL

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*] p \;:\; \nu ?.\, [a](p \wedge ?)$$

$$\Big\downarrow \; (M)_r$$

$$[a^*](p \wedge q) \,;\, \neg [a^*] p \;:\; p \wedge ?$$

$$\Big\downarrow \; (*)_\ell, (\wedge)_\ell$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a^*] p \;:\; p \wedge ?$$

$$(\neg *)_r$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg p \;:\; p \qquad\qquad p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*] p \;:\; ?$$

Cyclic Maehara for PDL

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*]p \;:\; [a^*][a]p$$

$$\Big\downarrow (M)_r$$

$$[a^*](p \wedge q) \,;\, \neg [a^*]p \;:\; p \wedge ?$$

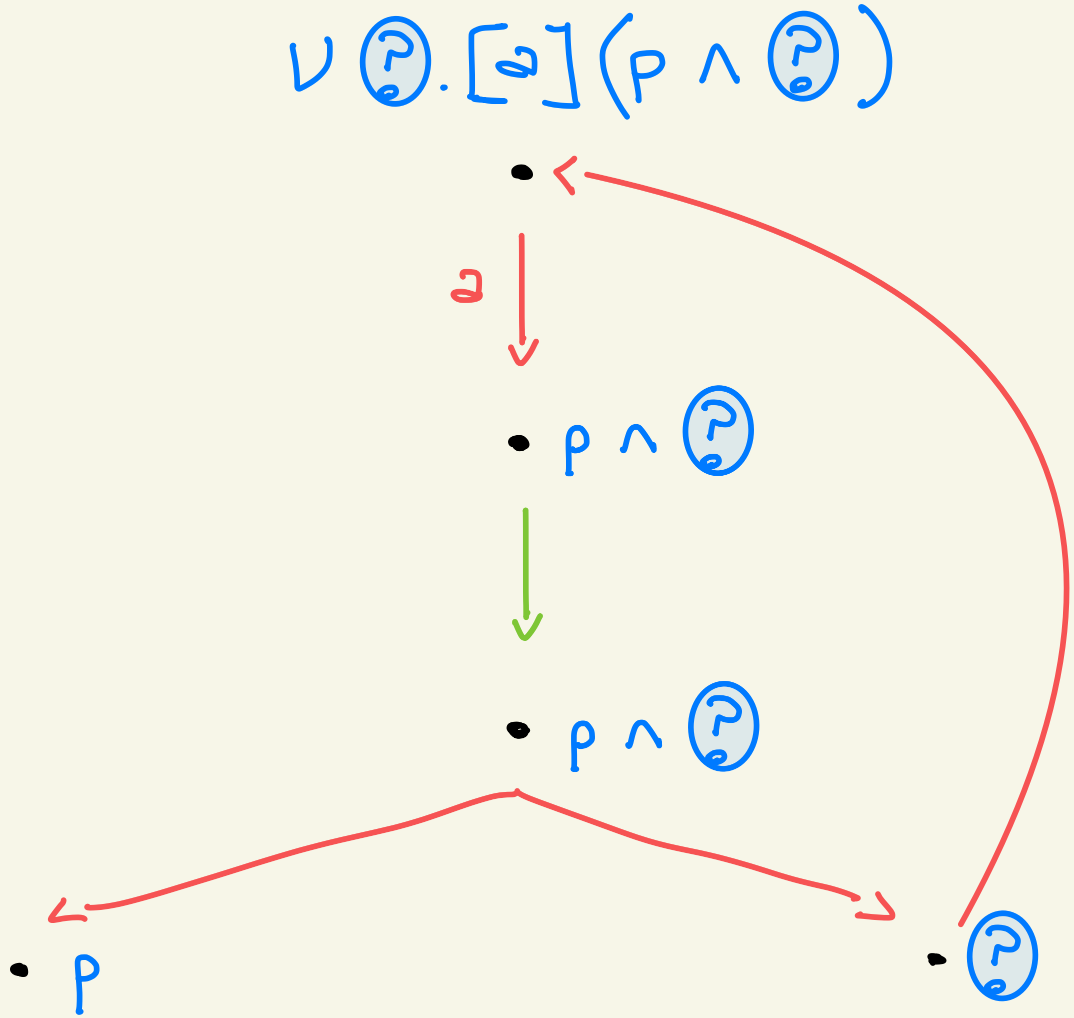
$$\Big\downarrow (*)_\ell, (\wedge)_\ell$$

$$p, q, [a][a^*](p \wedge q) \,;\, \neg [a^*]p \;:\; p \wedge ?$$

$$(\neg *)_r$$

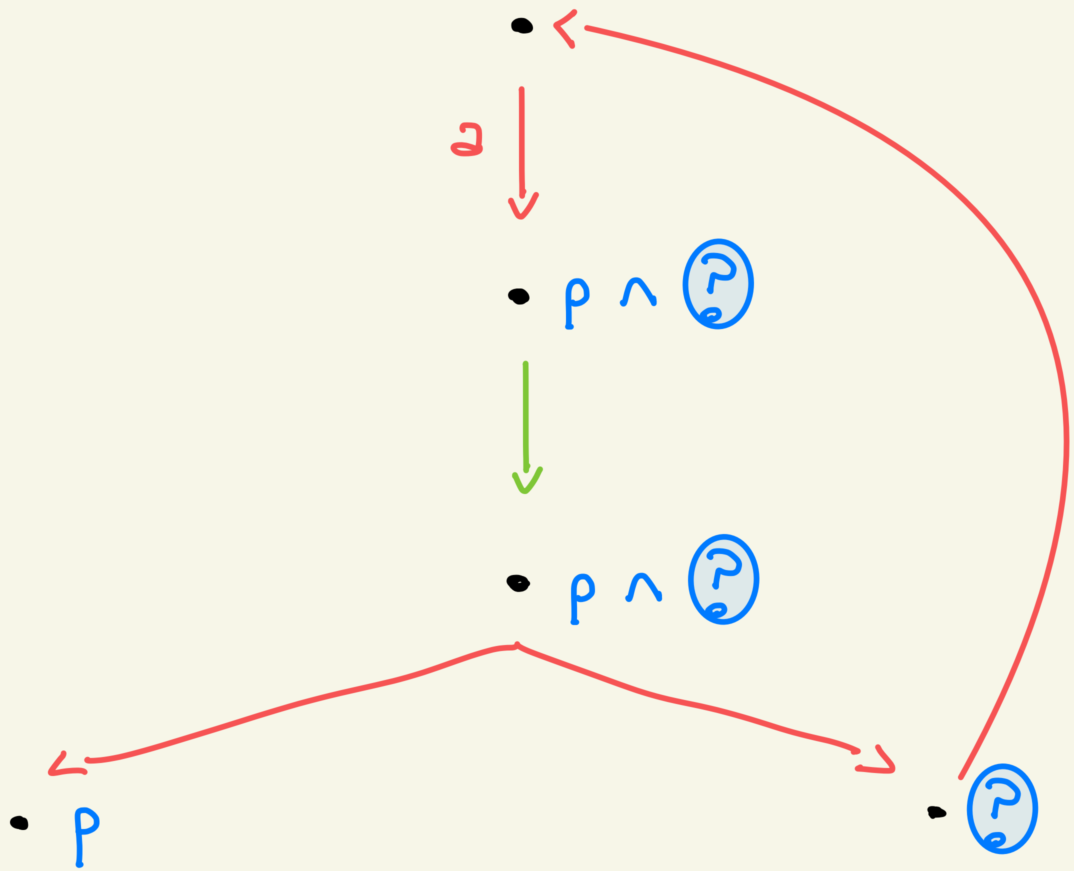
$$p, q, [a][a^*](p \wedge q) \,;\, \neg p \;:\; p \qquad\qquad p, q, [a][a^*](p \wedge q) \,;\, \neg [a][a^*]p \;:\; ?$$

Cyclic Maehara for PDL

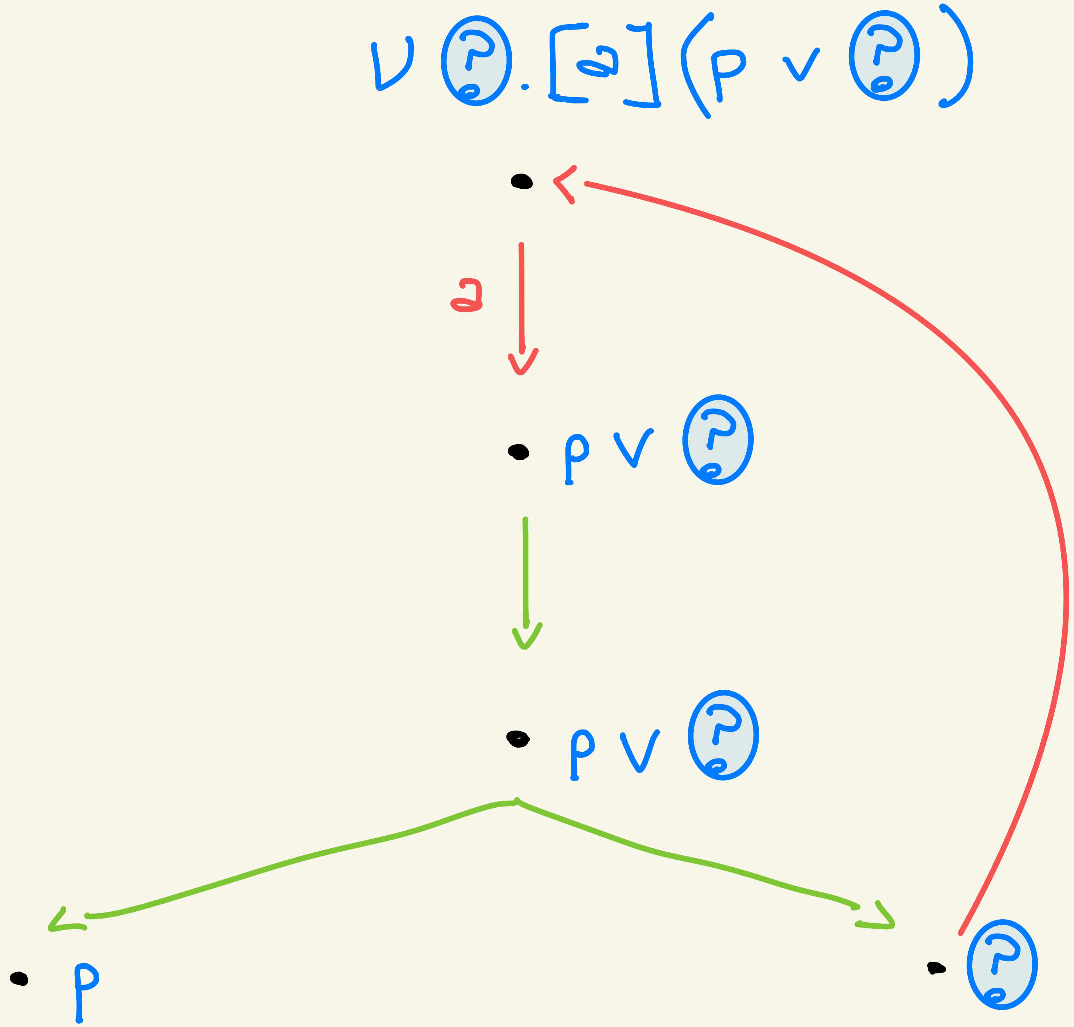


Cyclic Maehara for PDL

$\nu ?.[a](p \wedge ?) \equiv [a^*][a]p$

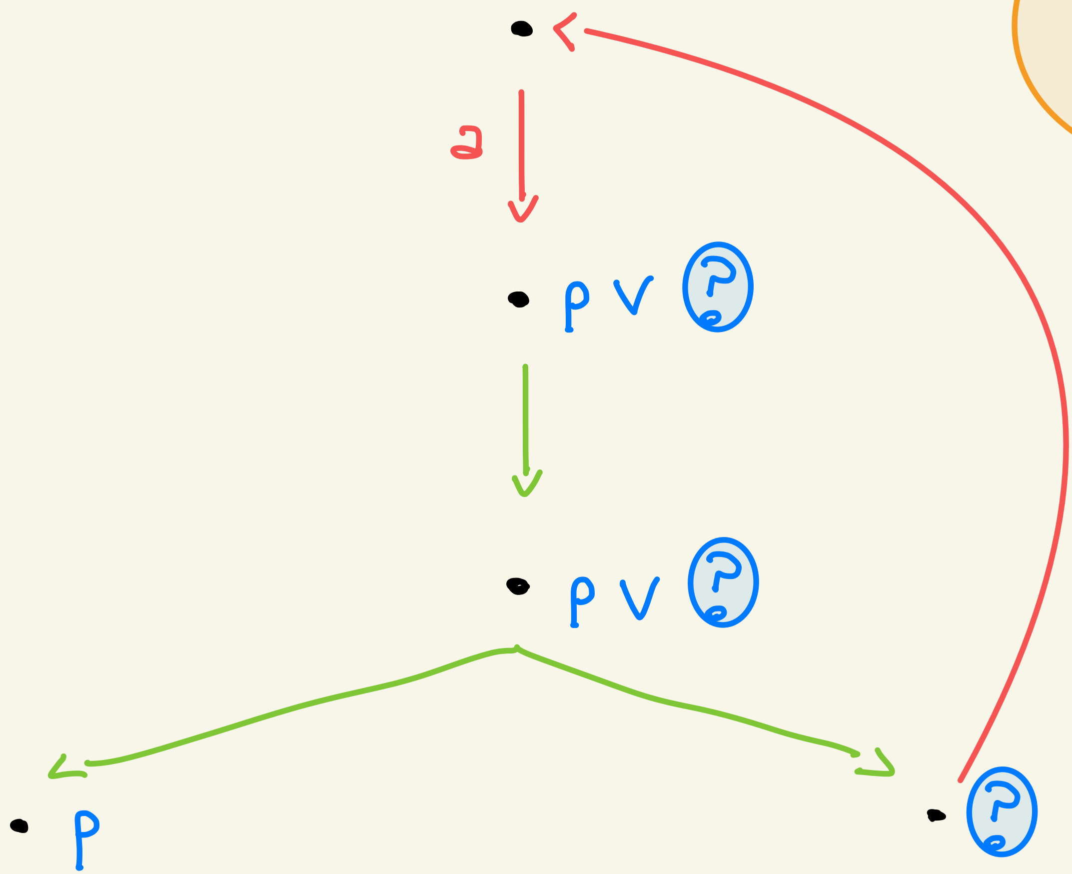


Cyclic Maehara for PDL



Cyclic Maehara for PDL

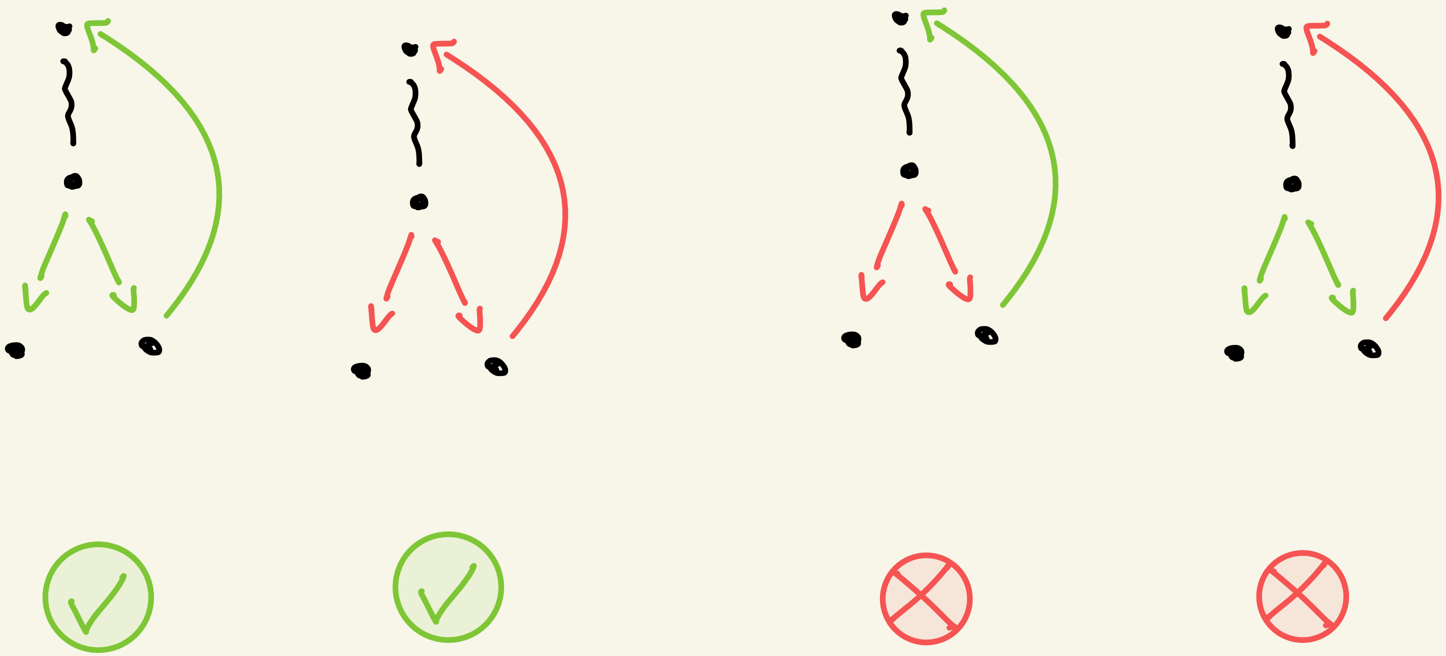
$\nu ?.[a](p \vee ?)$



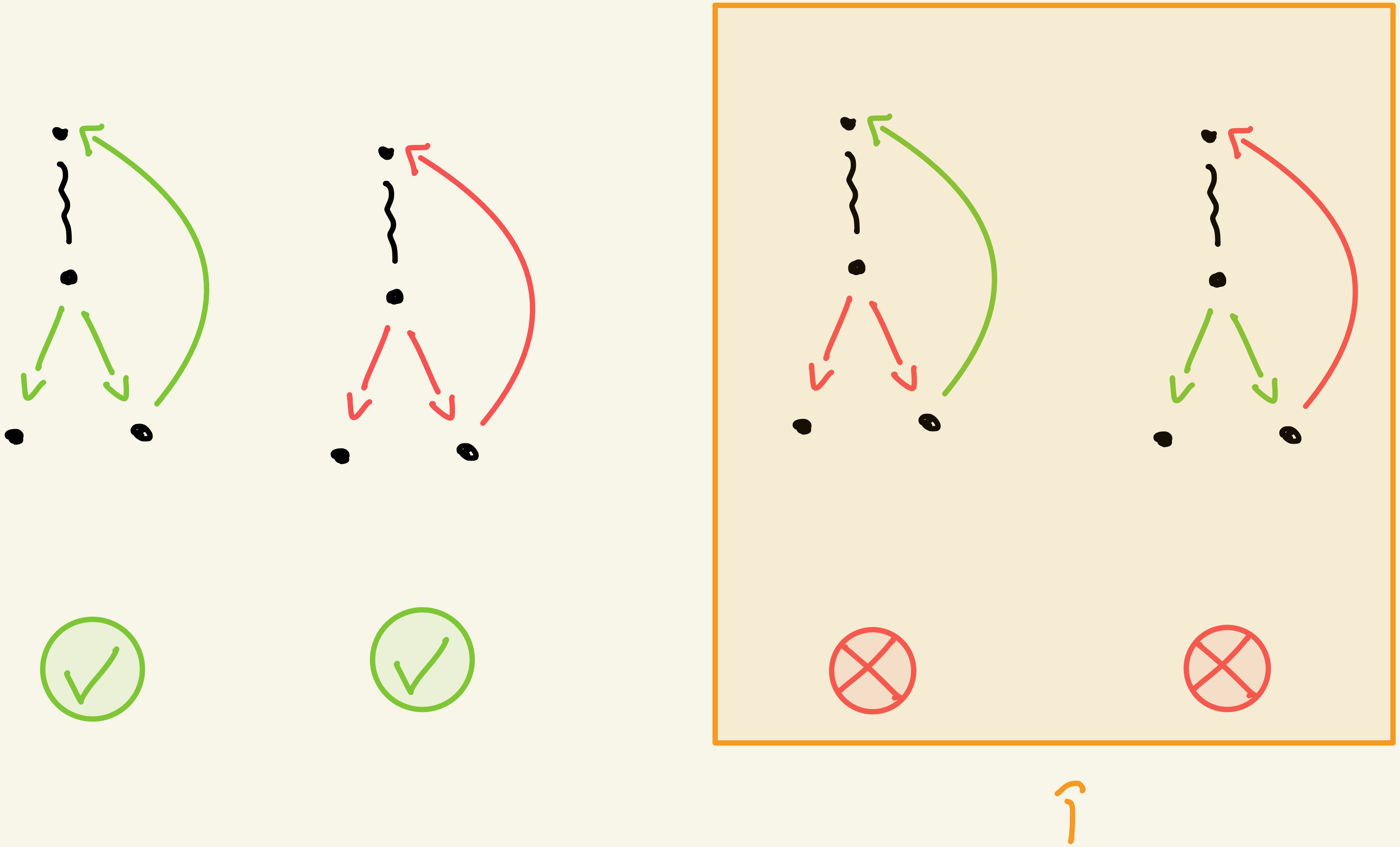
No solution in general in PDL

[CV14]

Cyclic Maehara for PDL

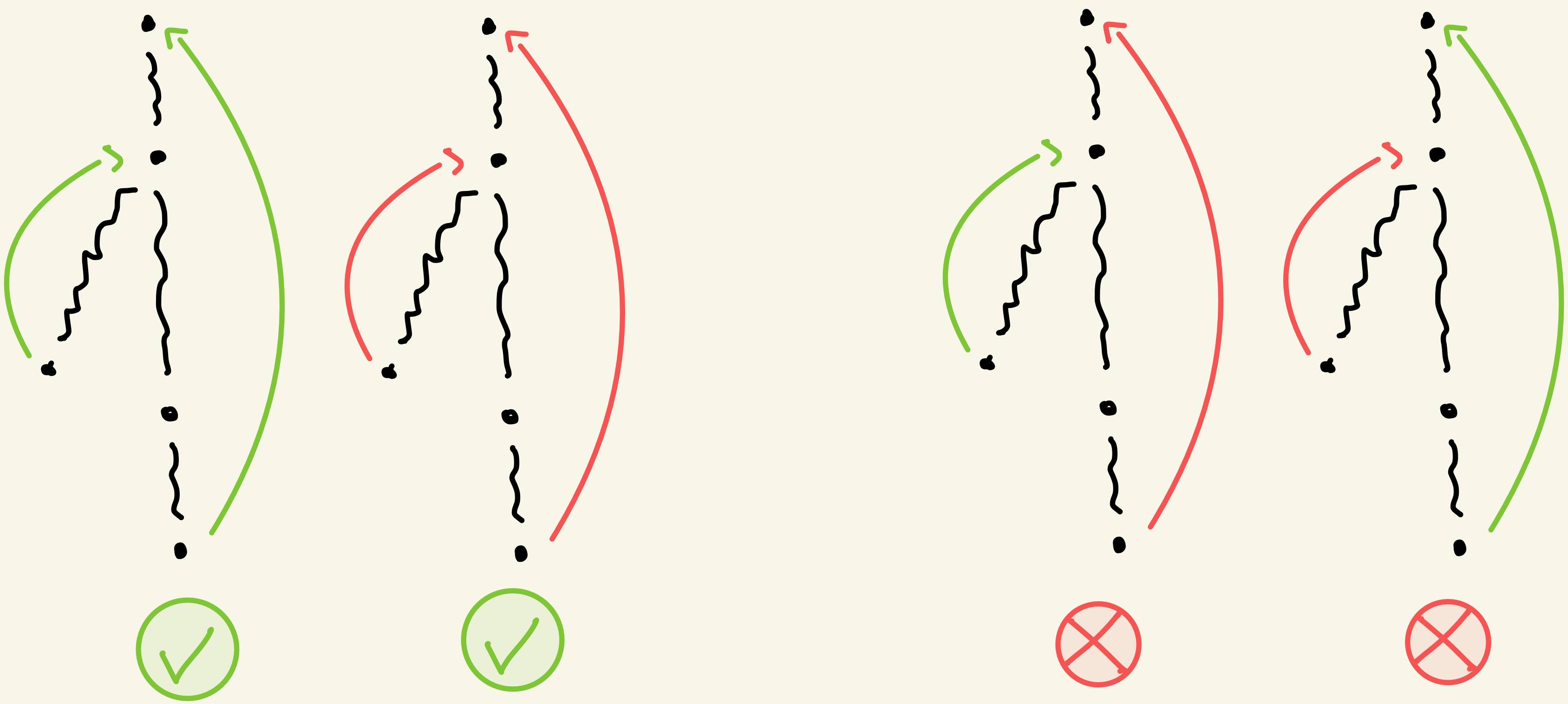


Cyclic Maehara for PDL

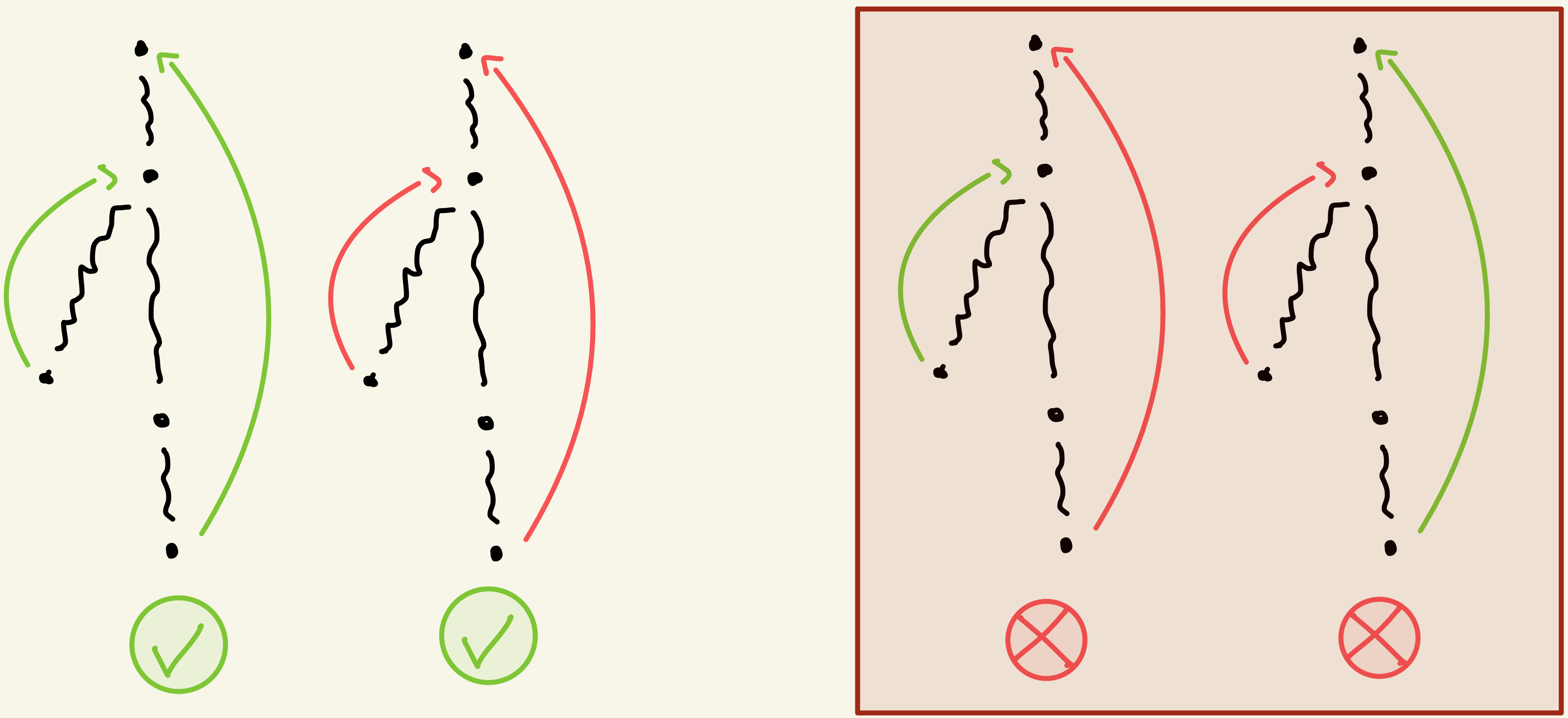


Quasitableaux

Cyclic Maehara for PDL

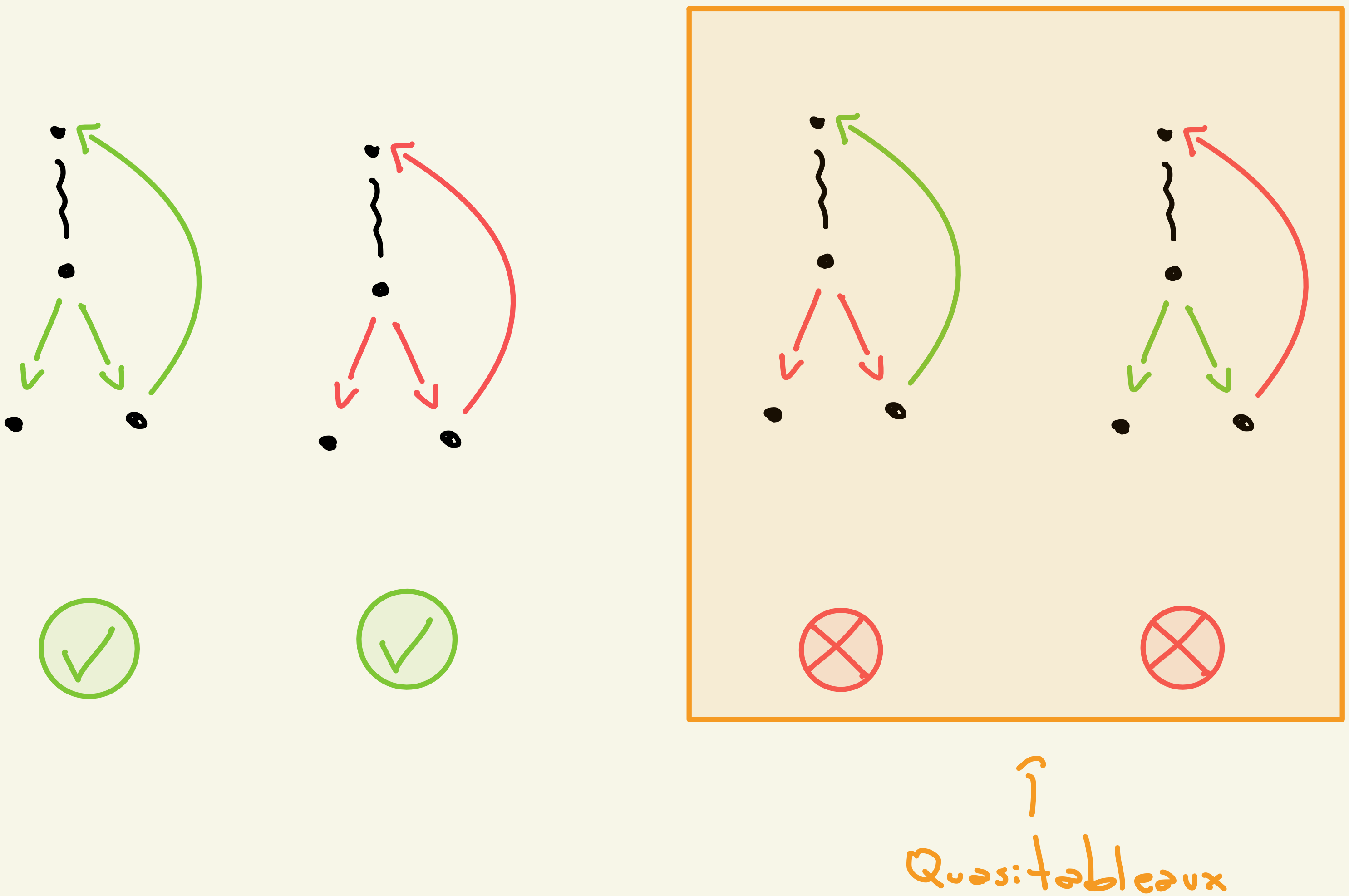


Cyclic Maehara for PDL



Loading mechanism

Cyclic Maehara for PDL



Quasitableaux

$$[a^*](p \vee \bot)\ ;\ \neg[a^*]p$$

$$\Big\downarrow (*)_l$$

$$p \vee \bot,\ [a][a^*](p \vee \bot)\ ;\ \neg[a^*]p$$

$$(\vee)_l$$

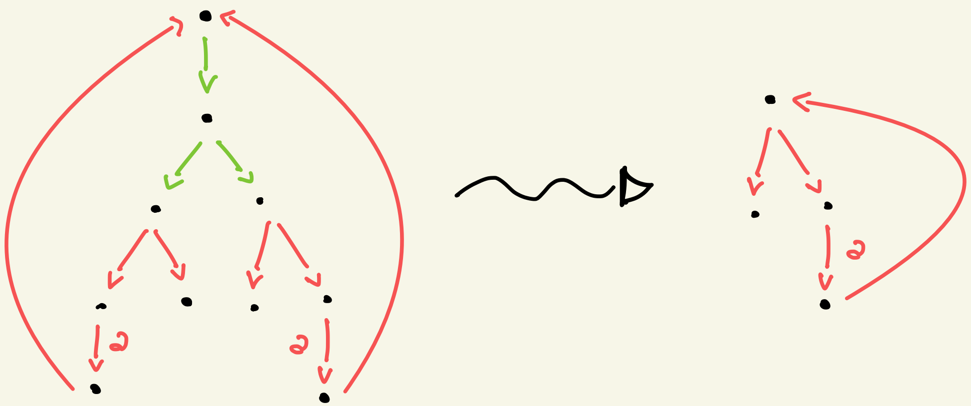
$$p,\ [a][a^*](p \vee \bot)\ ;\ \neg[a^*]p \qquad\qquad \bot,\ [a][a^*](p \vee \bot)\ ;\ \neg[a^*]p$$

$$(\neg *)_r \qquad\qquad (\neg *)_r$$

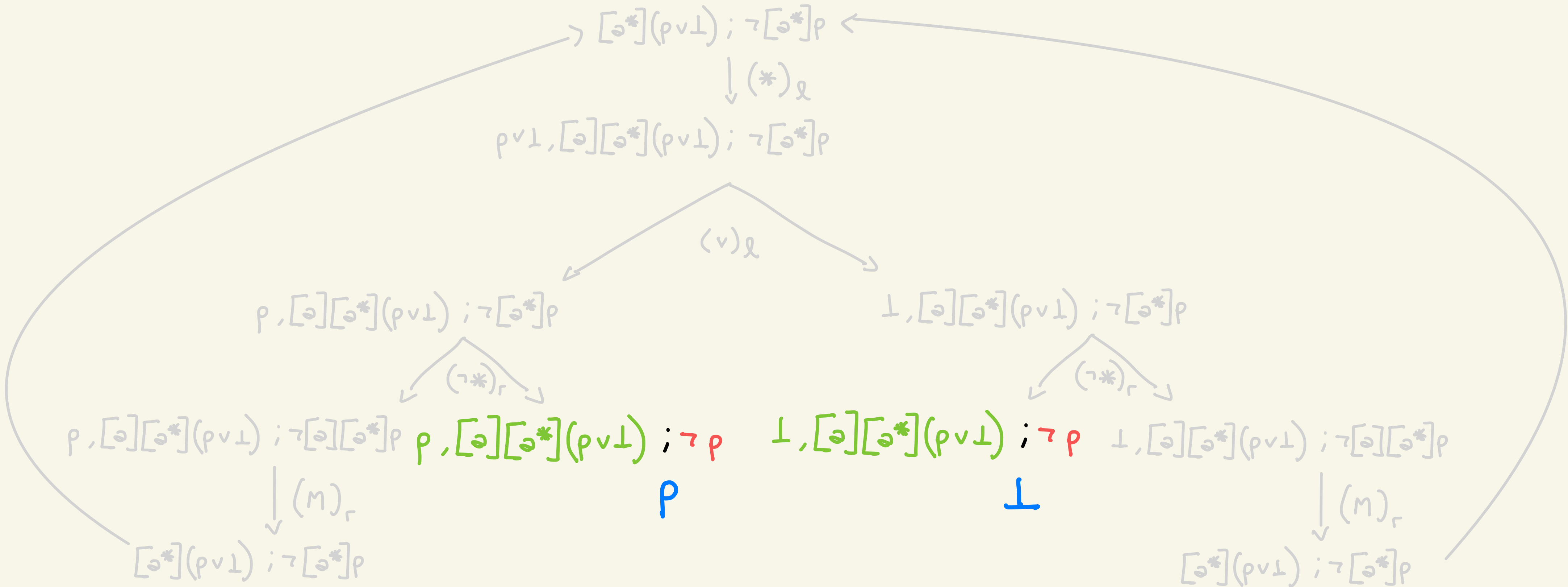
$$p,\ [a][a^*](p \vee \bot)\ ;\ \neg[a][a^*]p \qquad p,\ [a][a^*](p \vee \bot)\ ;\ \neg p \qquad \bot,\ [a][a^*](p \vee \bot)\ ;\ \neg p \qquad \bot,\ [a][a^*](p \vee \bot)\ ;\ \neg[a][a^*]p$$

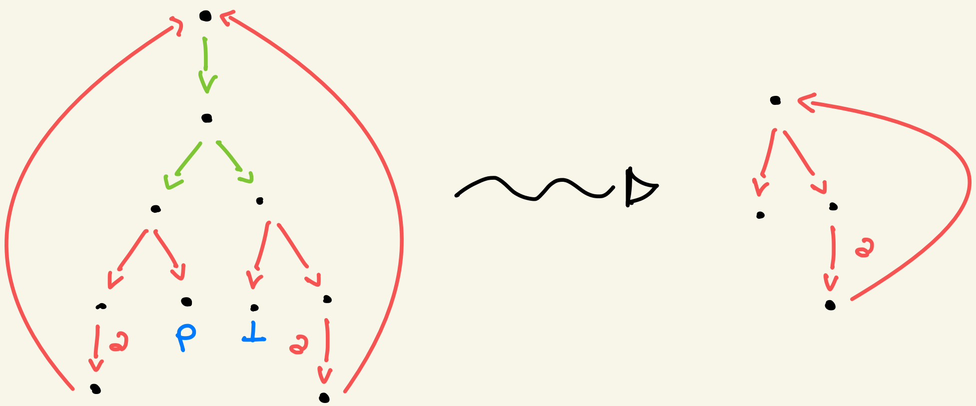
$$\Big\downarrow (M)_r \qquad\qquad \Big\downarrow (M)_r$$

$$[a^*](p \vee \bot)\ ;\ \neg[a^*]p \qquad\qquad [a^*](p \vee \bot)\ ;\ \neg[a^*]p$$



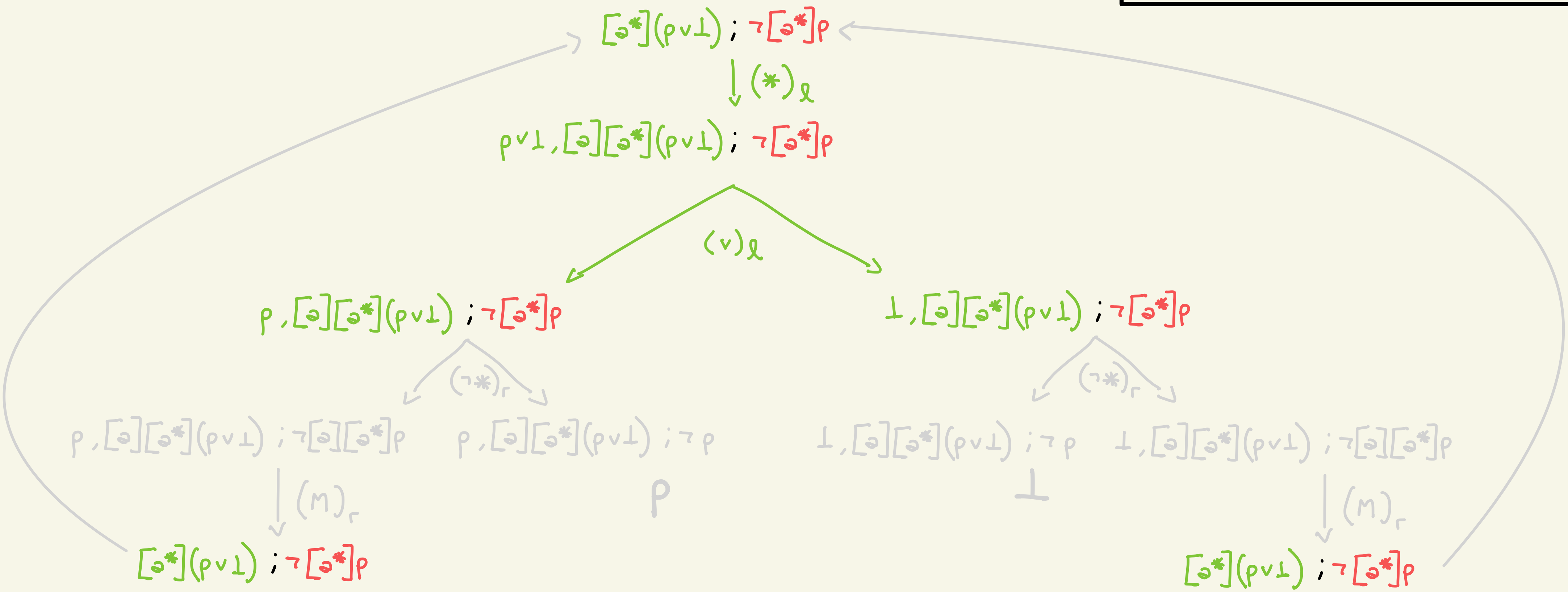
Quasitableaux





Quasitableaux

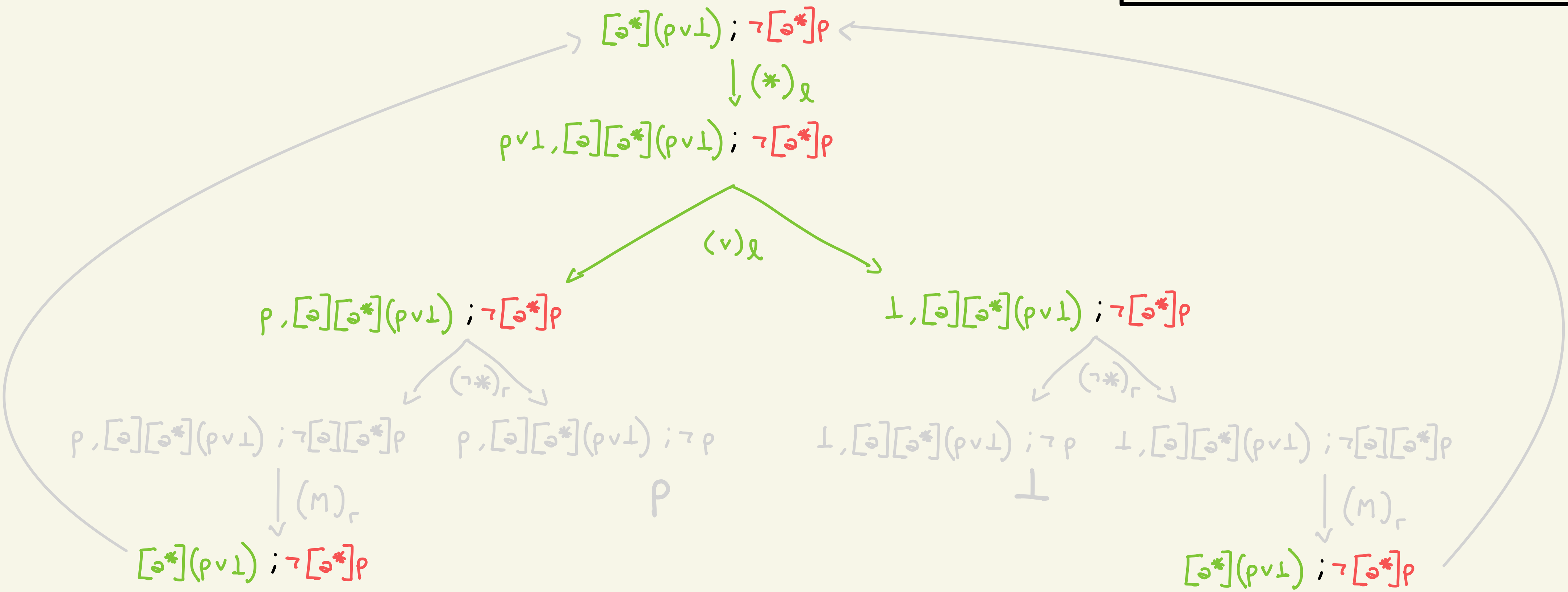
$\Delta_x = \neg[a^*]p$



$x : \rho_x ; \Delta_x$

Quasitableaux

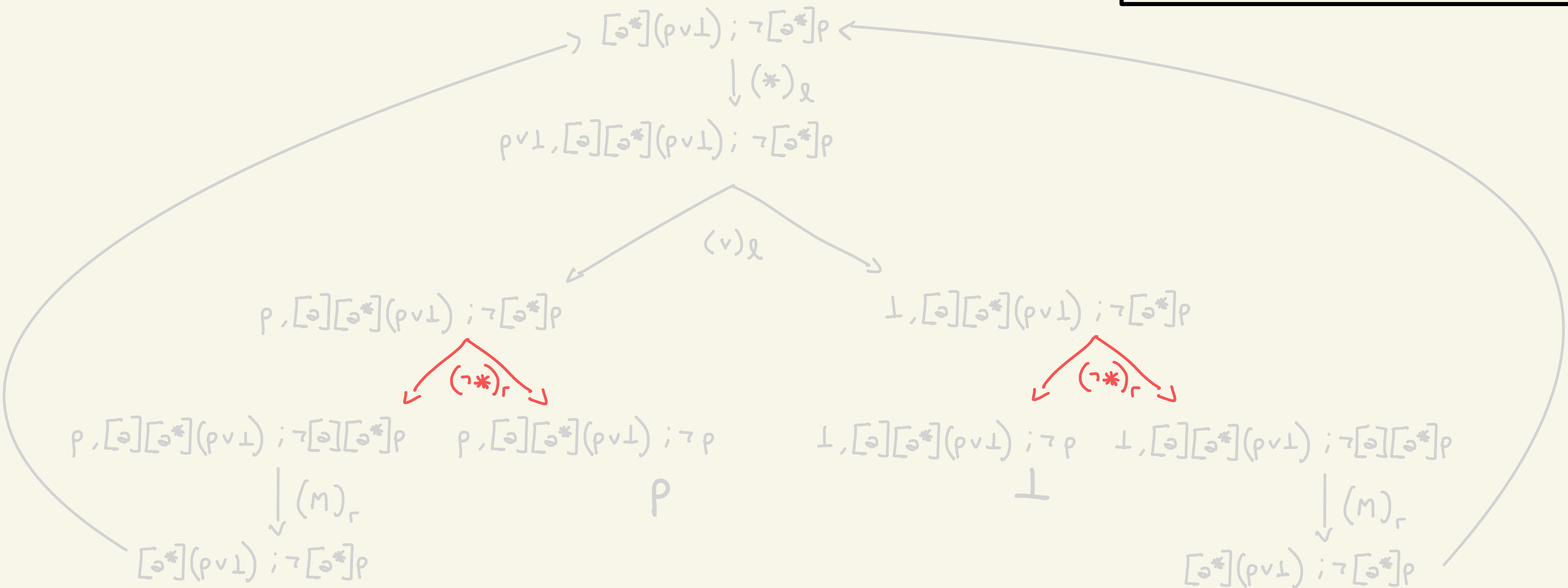
$\Delta_x = \neg[a^*]p$



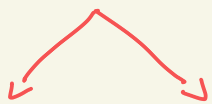
$$\boxed{\begin{array}{l} \big[[a^*](p\vee\bot)\big] \vee \\ \big[(p\vee\bot)\wedge[a][a^*](p\vee\bot)\big] \vee \\ \big[p\wedge[a][a^*](p\vee\bot)\big] \vee \\ \big[[a^*](p\vee\bot)\big] \vee \\ \big[\bot\wedge[a][a^*](p\vee\bot)\big] \vee \\ \big[[a^*](p\vee\bot)\big] \end{array}} = \rho_x \; ; \; \neg[a^*]p$$

Quasitableaux

$\Delta_z = \neg[a][a^*]p$

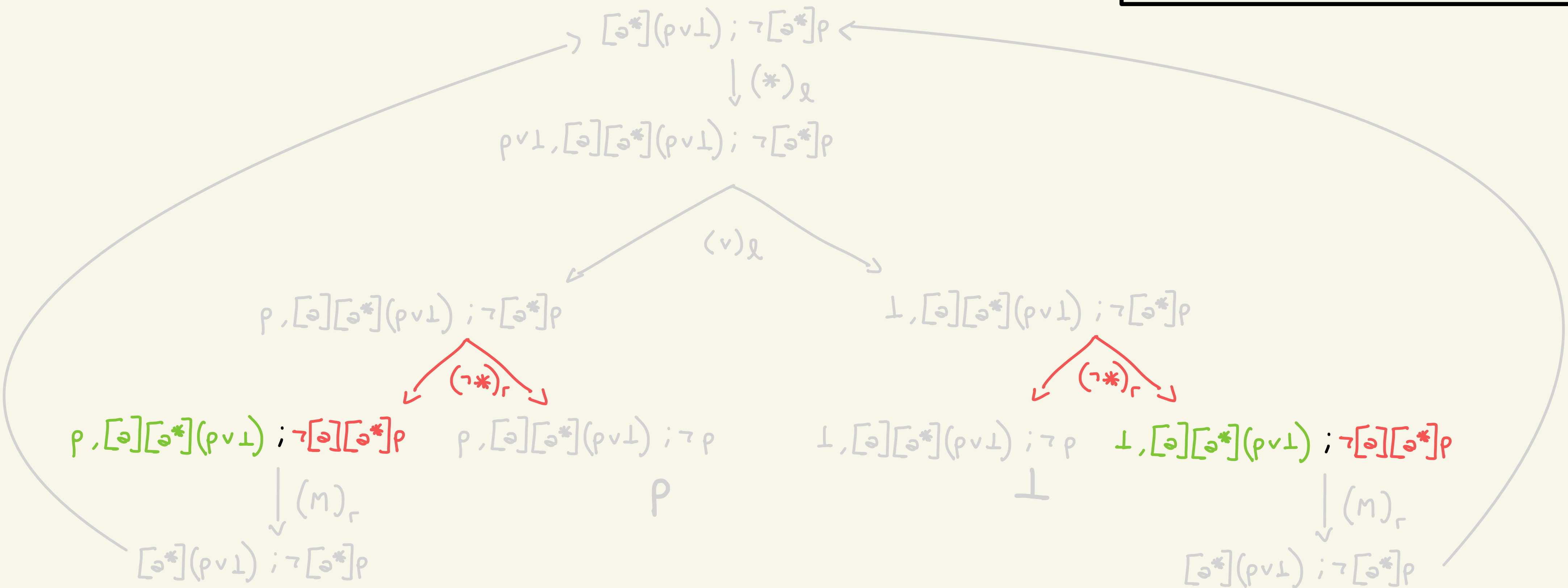


$x : p_x \; ; \; \neg[a^*]p$



Quasitableaux

$\Delta_z = \neg[a][a^*]p$

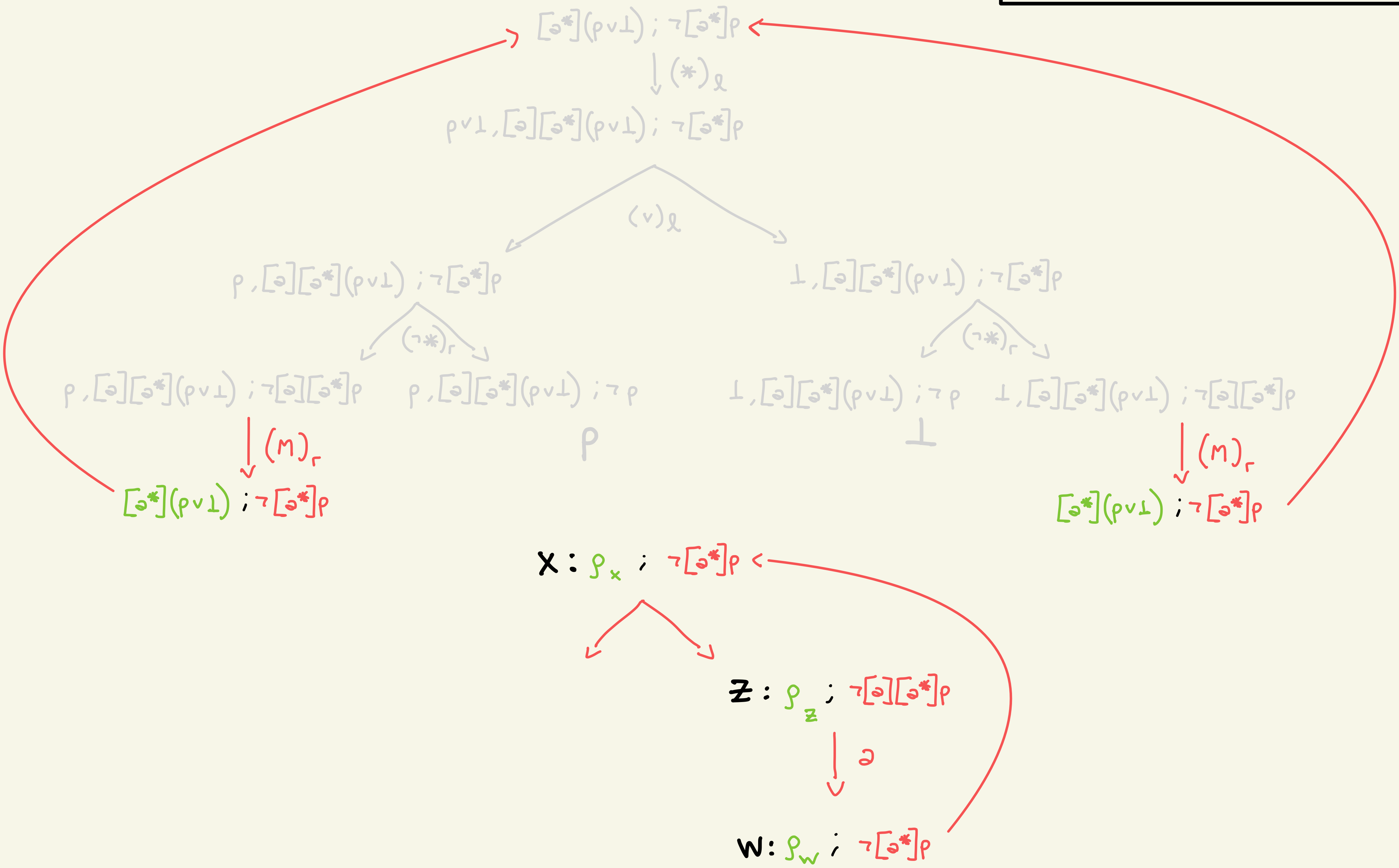


$x : p_x \,;\, \neg[a^*]p$

$z : p_z \,;\, \neg[a][a^*]p$

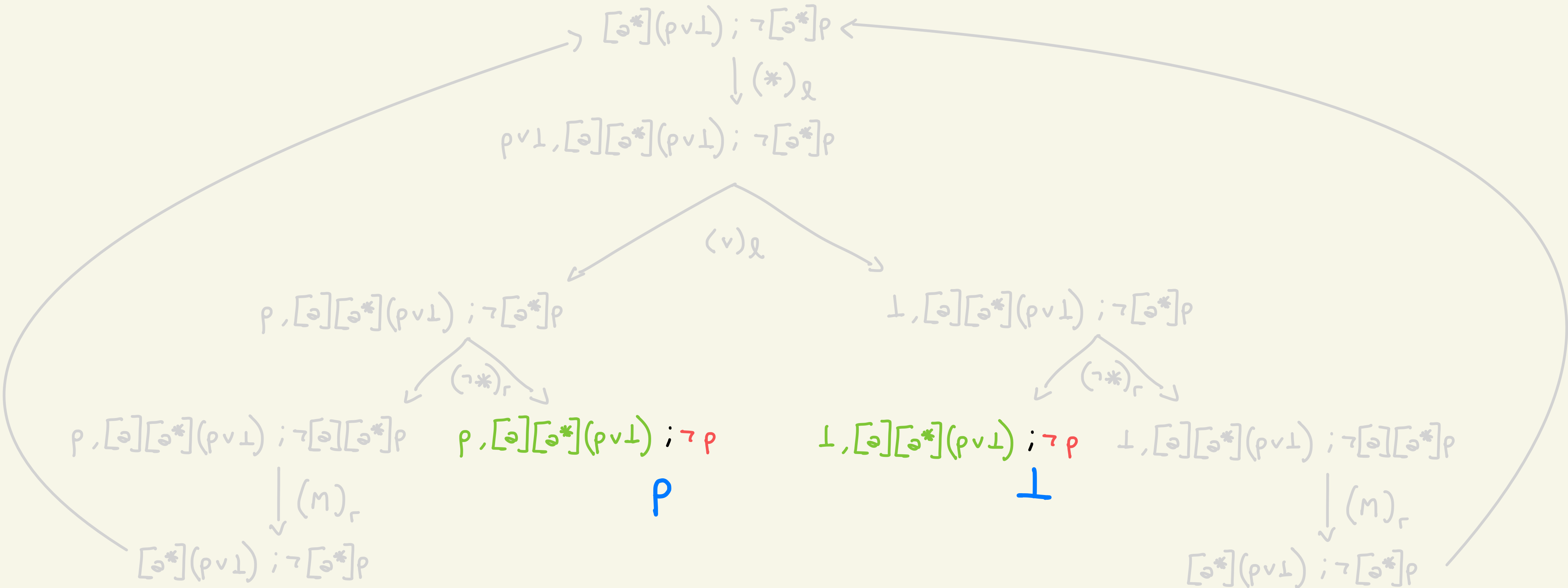
Quasitableaux

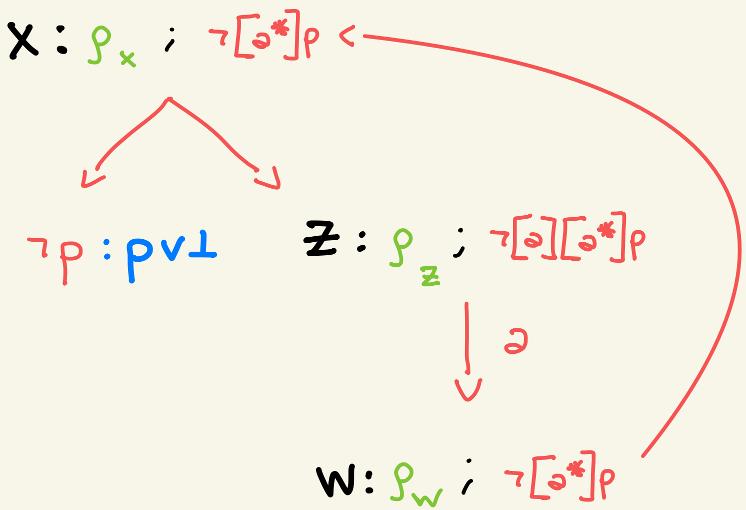
$\Delta_w = \neg[a^*]p$



Quasitableaux

$\Delta_y = \neg p$





Quasitableaux

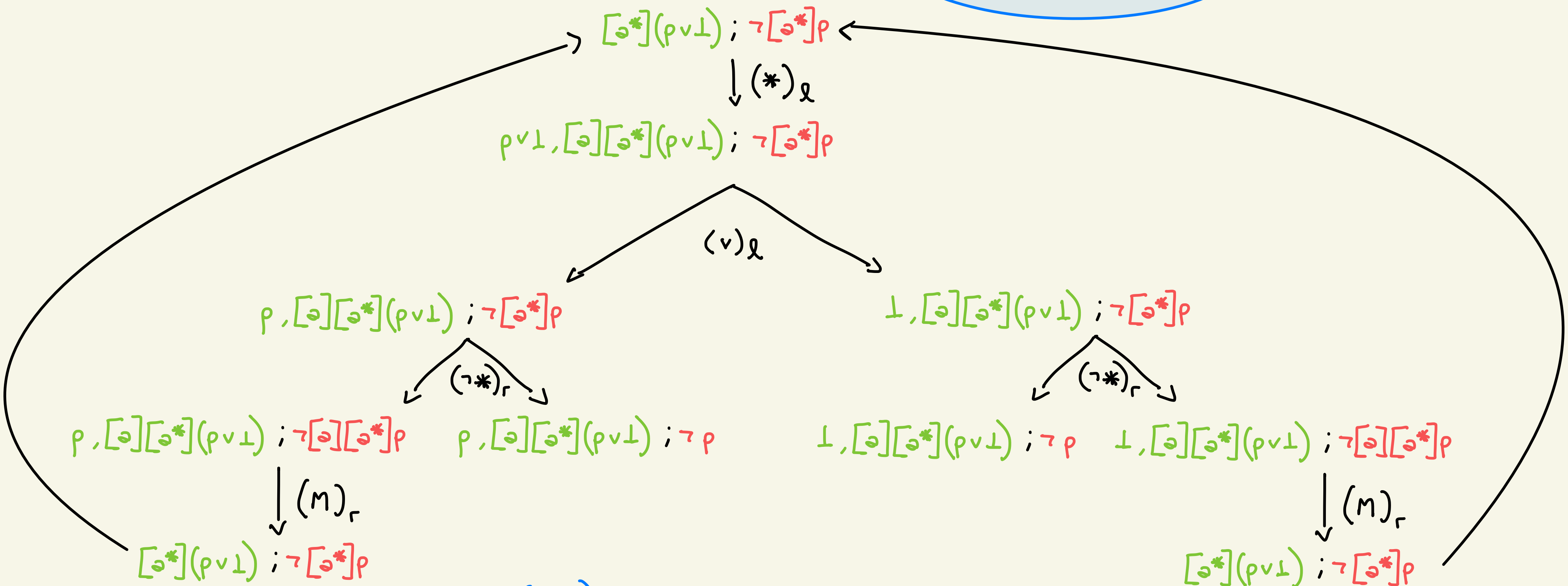
$$
\begin{tikzcd}[column sep=small]
& x : \rho_x \,;\, \neg[a^*]p : \nu X.\,[a]X \wedge (p \vee \bot) \arrow[dl] \arrow[dr] & \\
y : \rho_y \,;\, \neg p : p \vee \bot & & z : \rho_z \,;\, \neg[a][a^*]p : [a]X \arrow[d, "a"] \\
& & w : \rho_w \,;\, \neg[a^*]p : X \arrow[uul, bend right=60]
\end{tikzcd}
$$

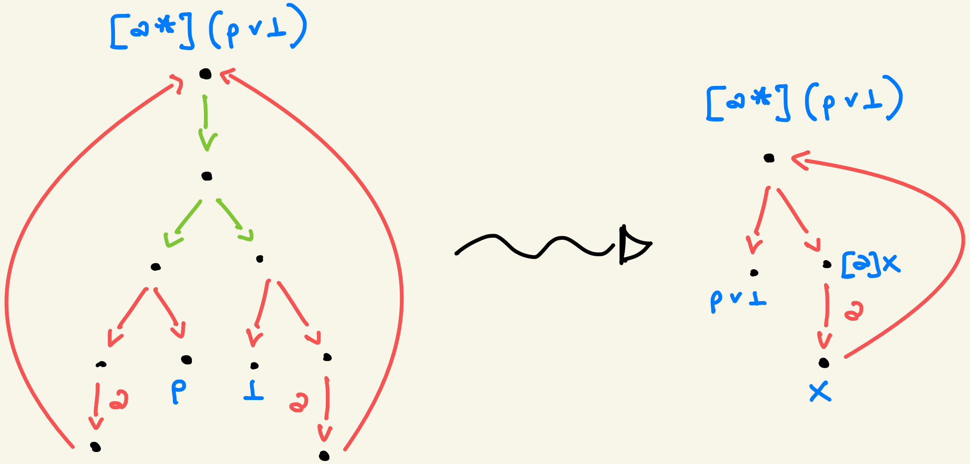
Quasitableaux

$$\begin{tikzcd}
 & x : \rho_x \,;\, \neg[a^*]p : [a^*](p \vee \bot) \arrow[dl] \arrow[dr] & \\
y : \rho_y \,;\, \neg p : p \wedge \bot & & z : \rho_z \,;\, \neg[a][a^*]p : [a]x \arrow[d, "a"] \\
 & & w : \rho_w \,;\, \neg[a^*]p : x \arrow[uul, bend right]
\end{tikzcd}$$

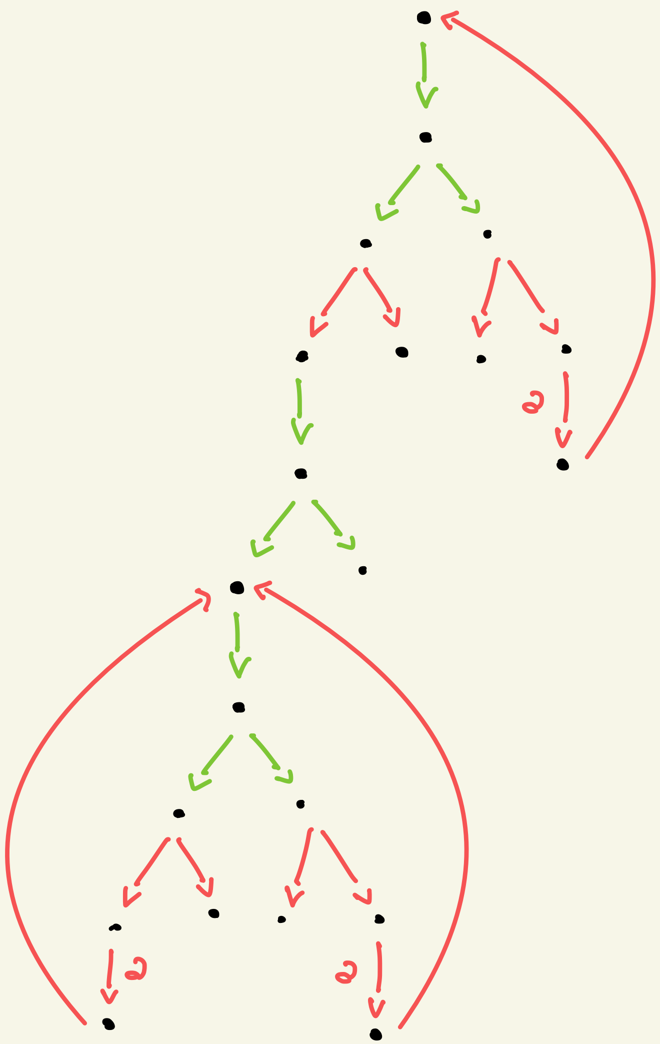
Quasitableaux

$[a^*](p \vee \bot)$

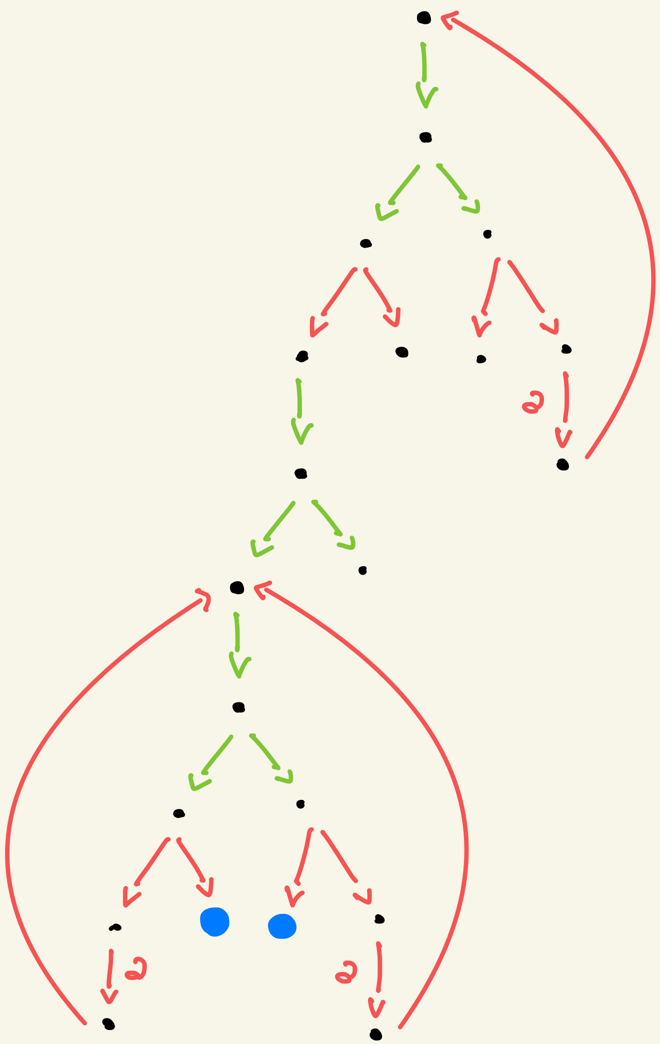




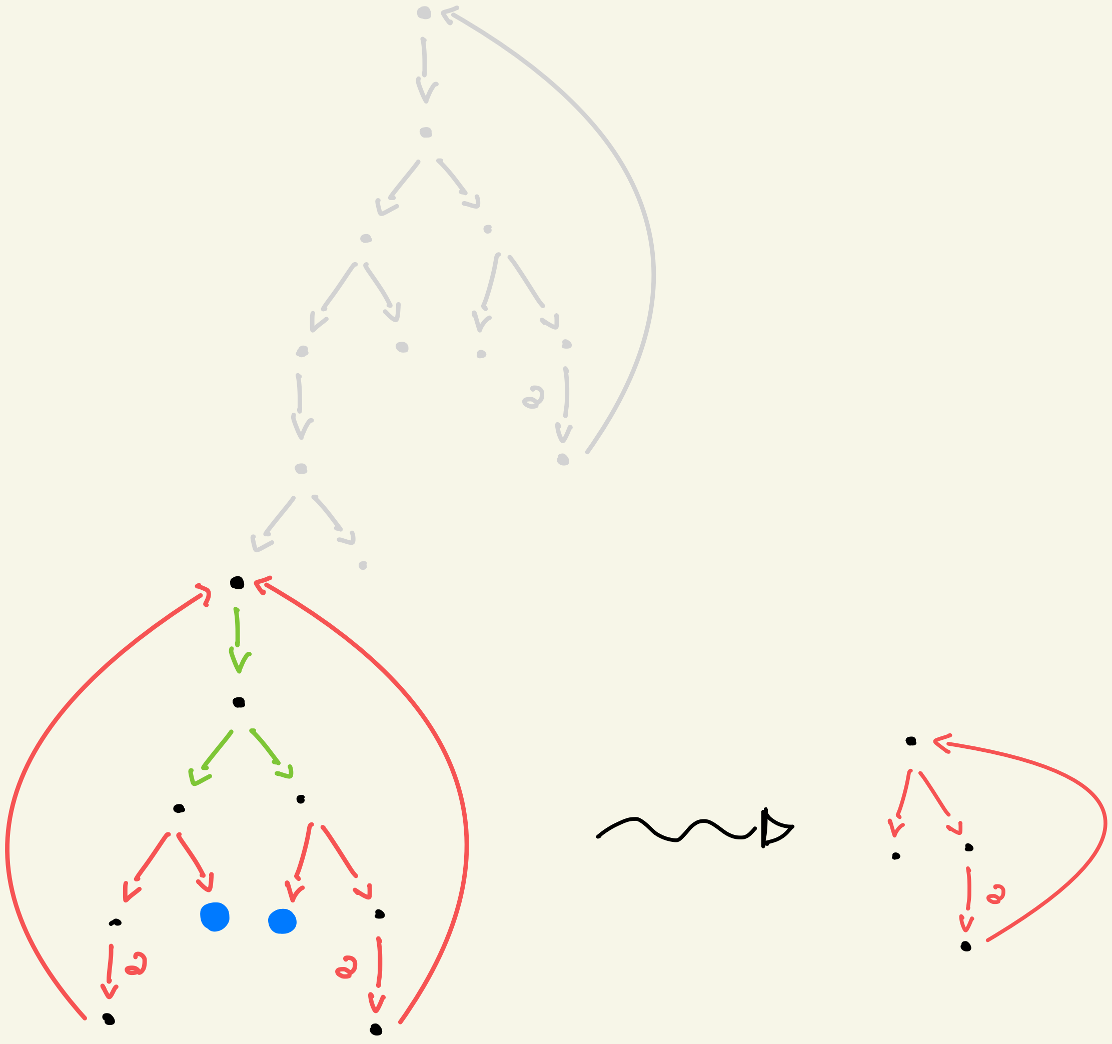
Putting it all together



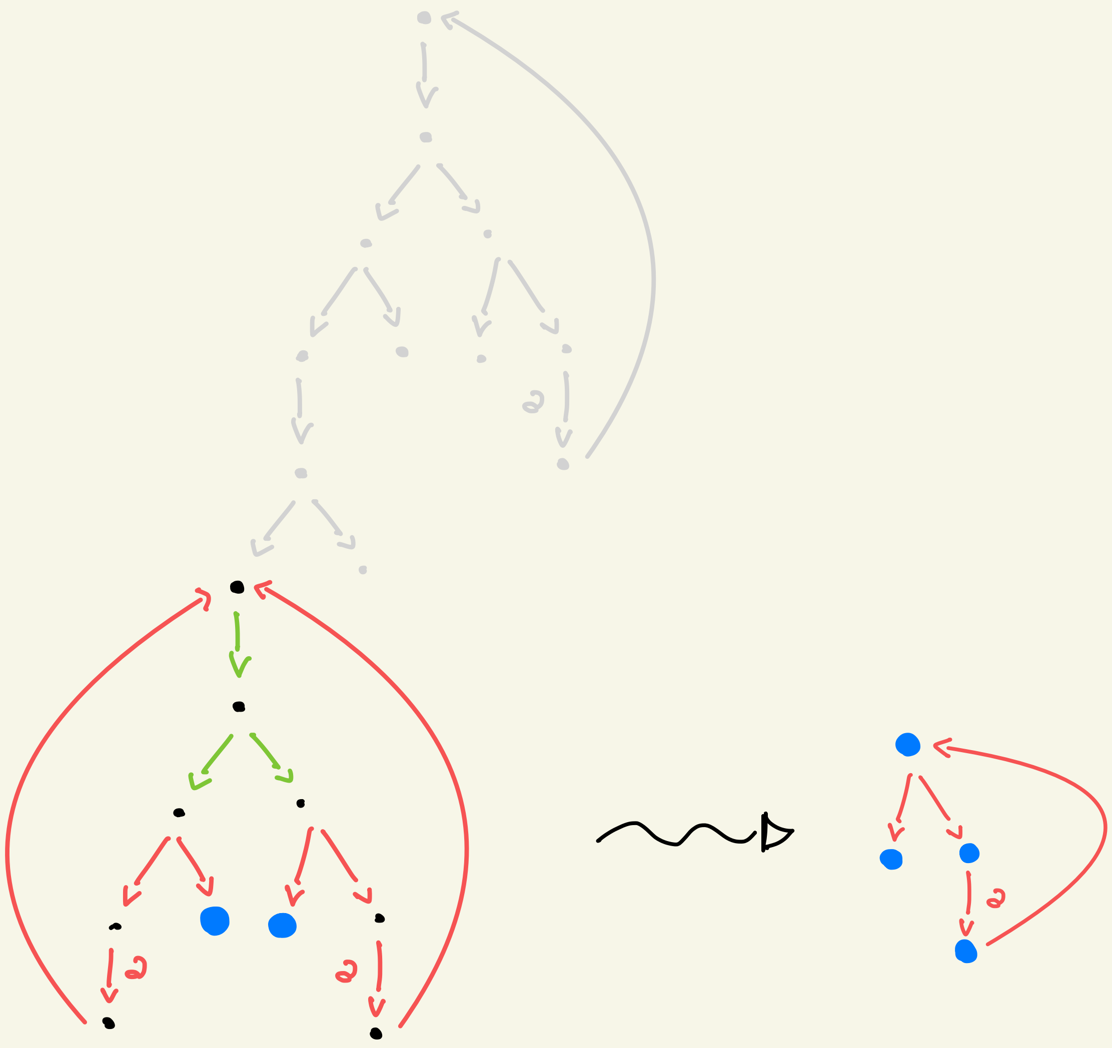
Putting it all together



Putting it all together



Putting it all together



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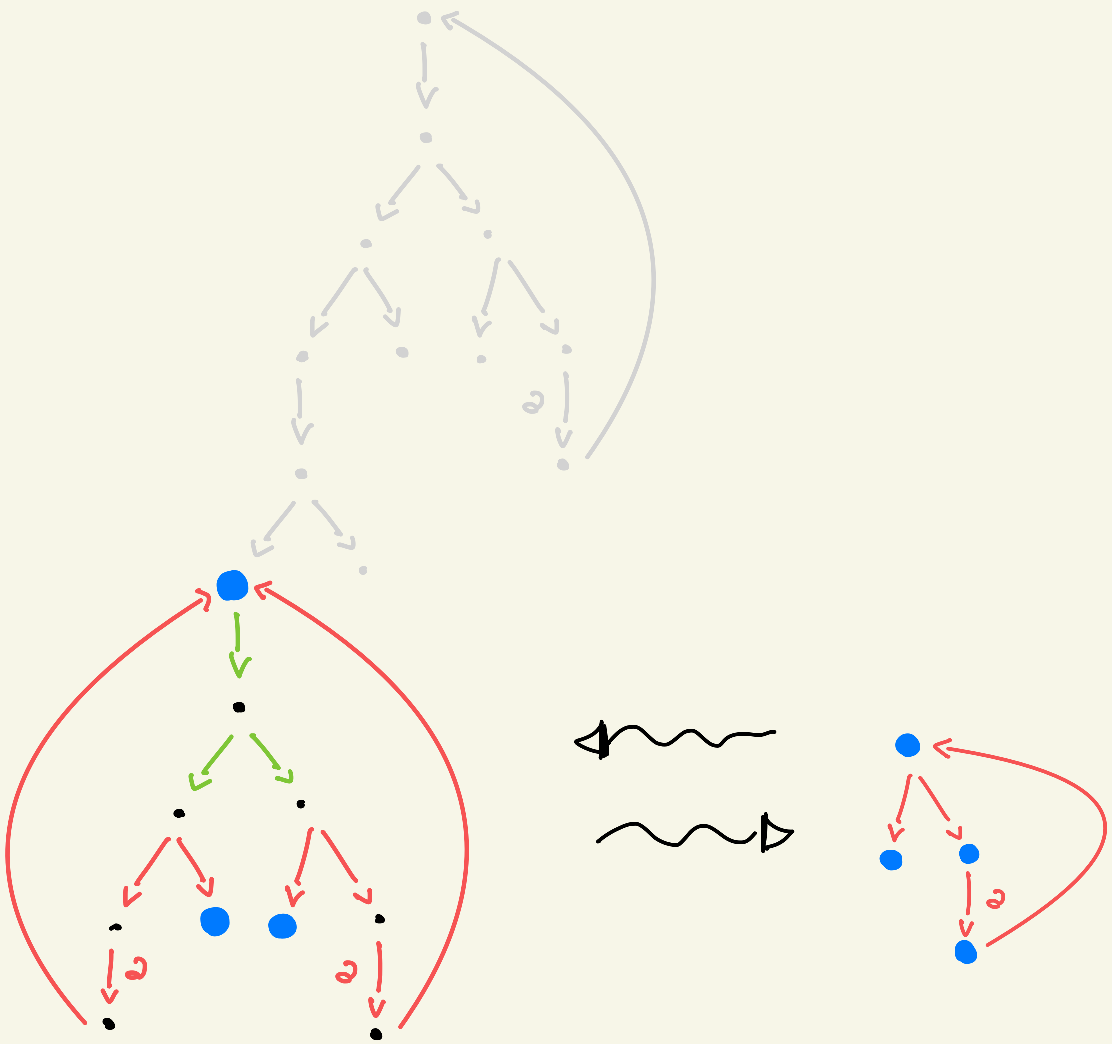

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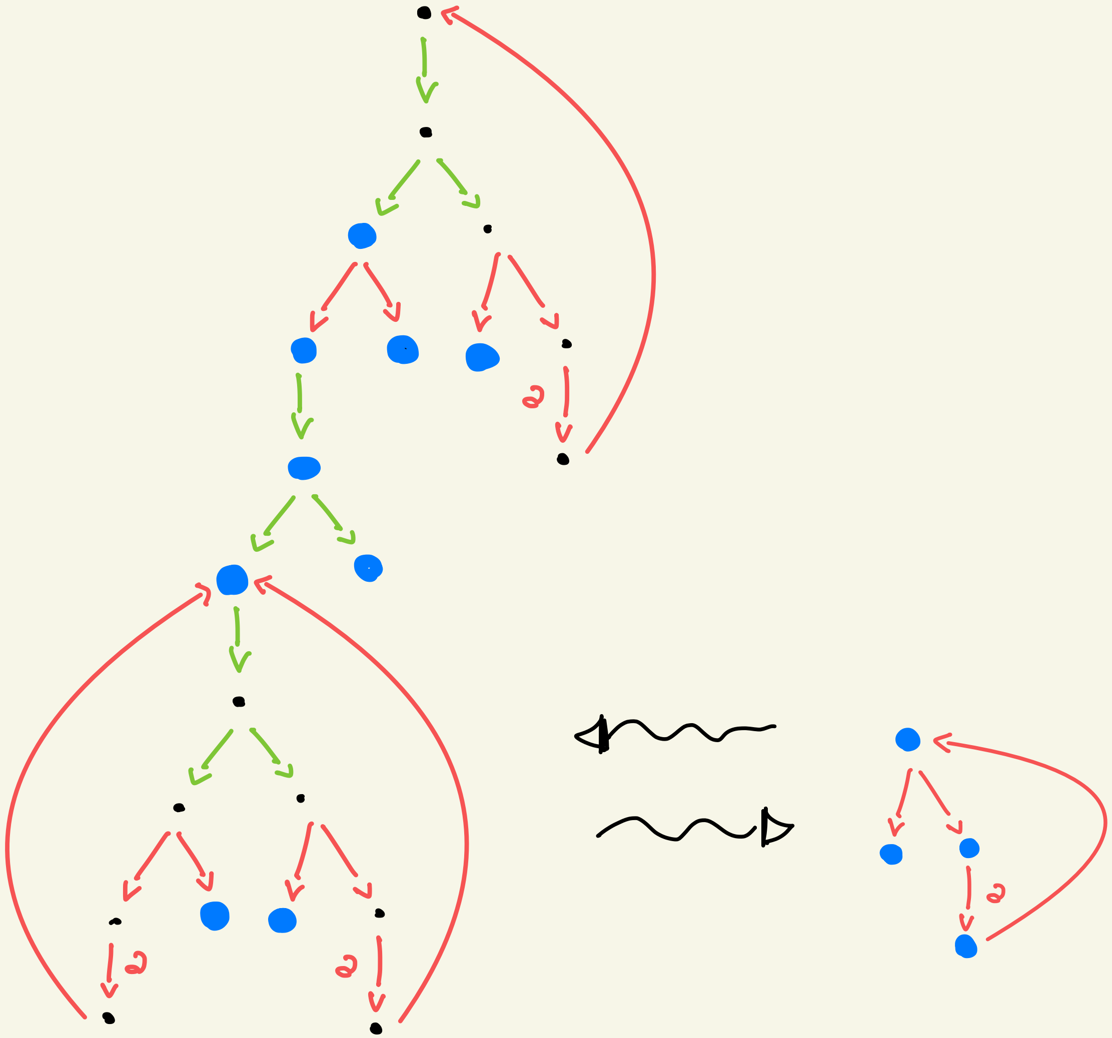

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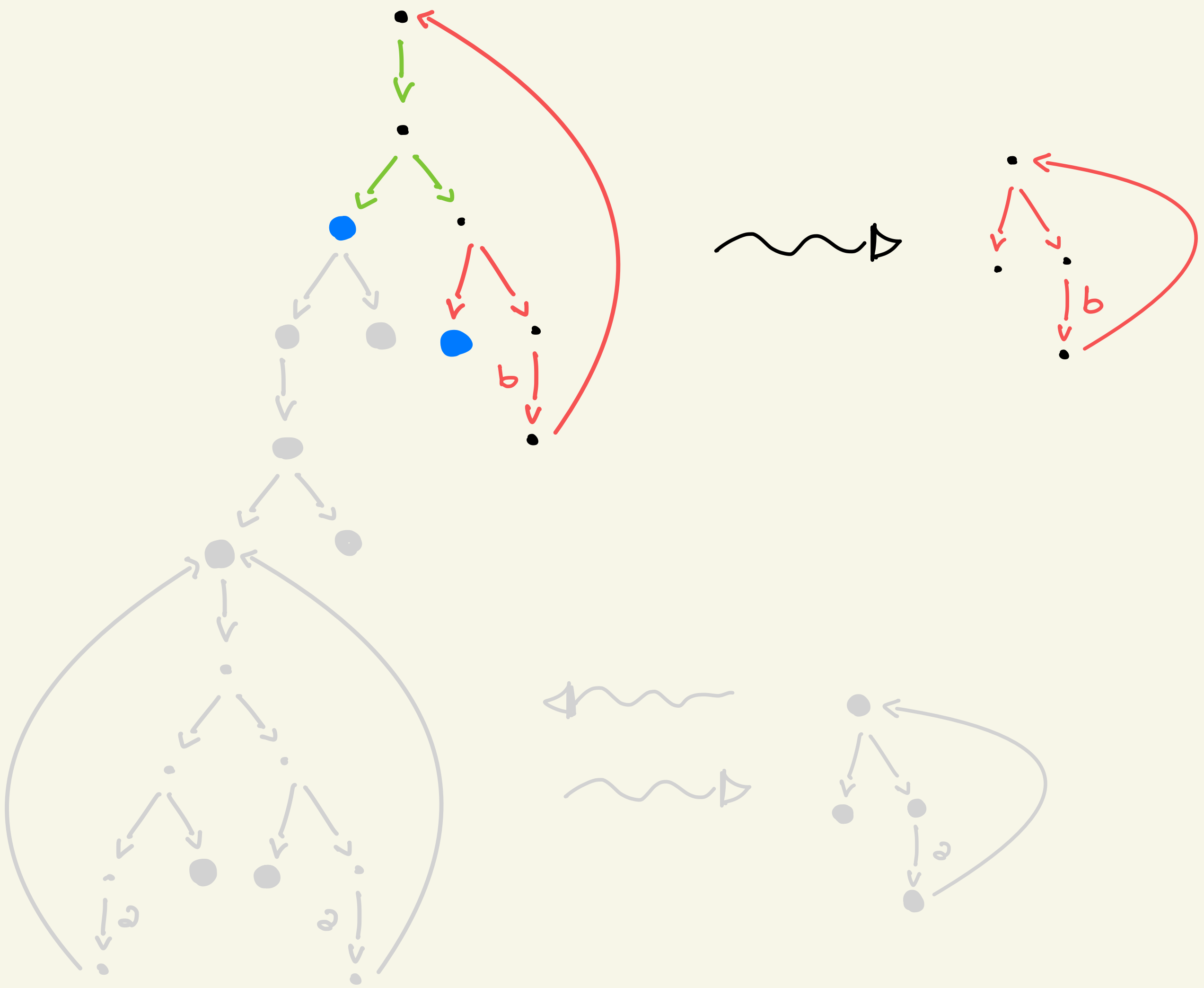
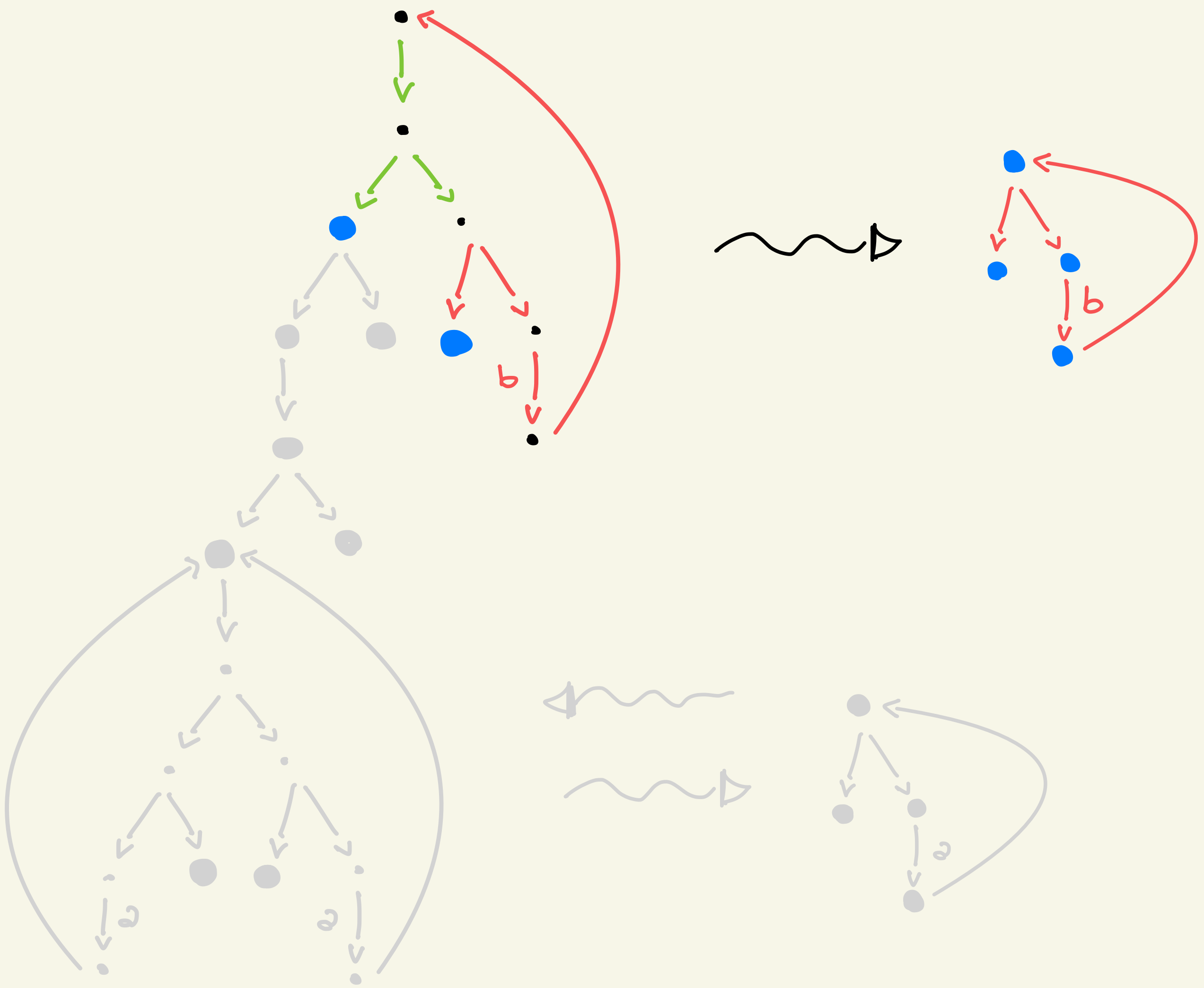
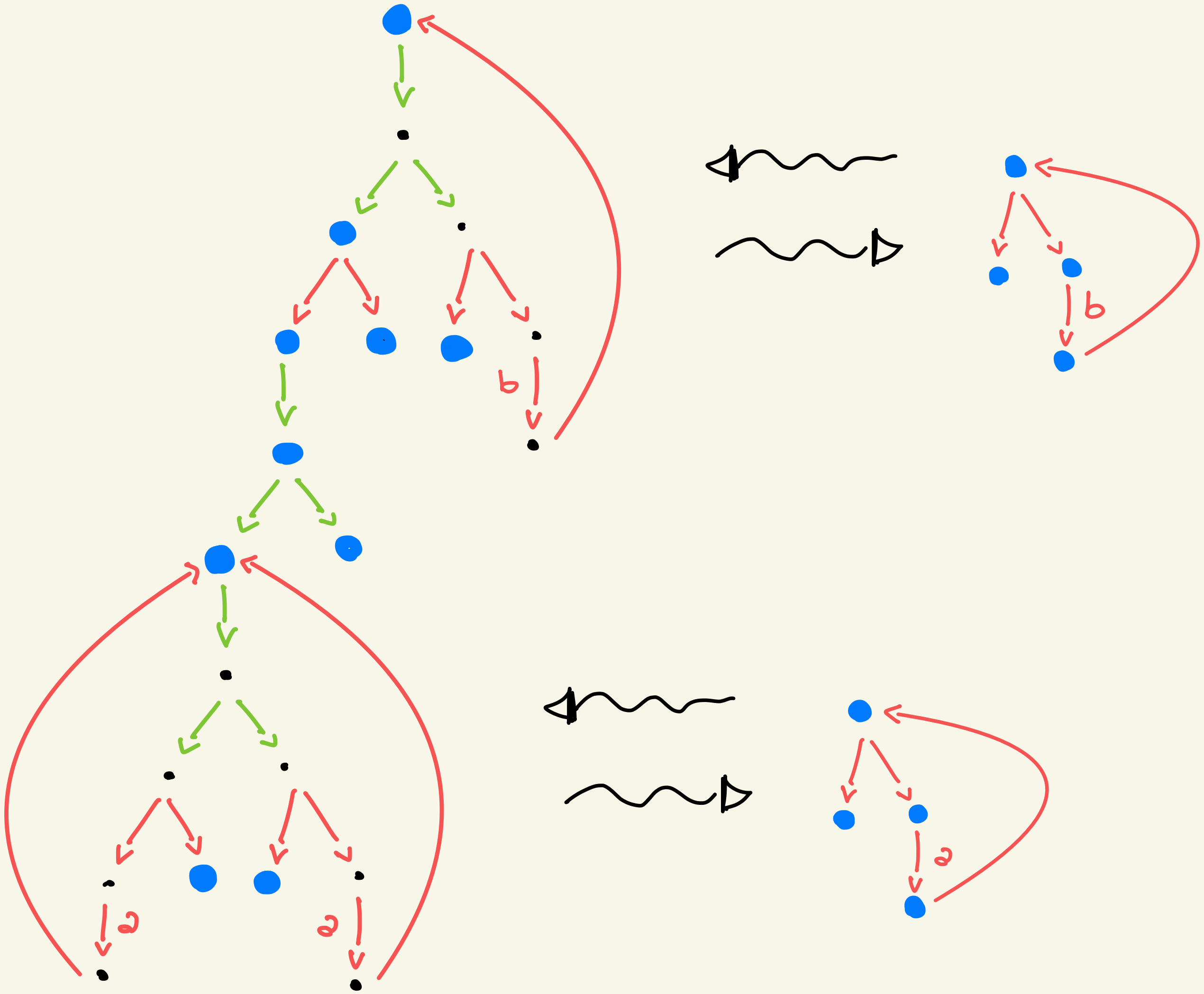


Putting it all together



Putting it all together



The PDL Tableau System

$$(\neg)\ \frac{\Delta, \neg\neg\varphi}{\Delta, \varphi} \qquad (\wedge)\ \frac{\Delta, \varphi \wedge \psi}{\Delta, \varphi, \psi} \qquad (\neg\wedge)\ \frac{\Delta, \neg(\varphi \wedge \psi)}{\Delta, \neg\varphi \ \mid\ \Delta, \neg\psi}$$

$$(\Box)\ \frac{\Delta, [\alpha]\varphi}{\left\{\Delta, \Gamma \mid \Gamma \in \mathsf{unfold}_{\Box}(\alpha, \varphi)\right\}} \quad \alpha \text{ non-atomic}$$

$$(\Diamond)\ \frac{\Delta, \neg[\alpha]\varphi}{\{\Delta, \Gamma \mid \Gamma \in \mathsf{unfold}_{\Diamond}(\alpha, \varphi)\}} \quad \alpha \text{ non-atomic}$$

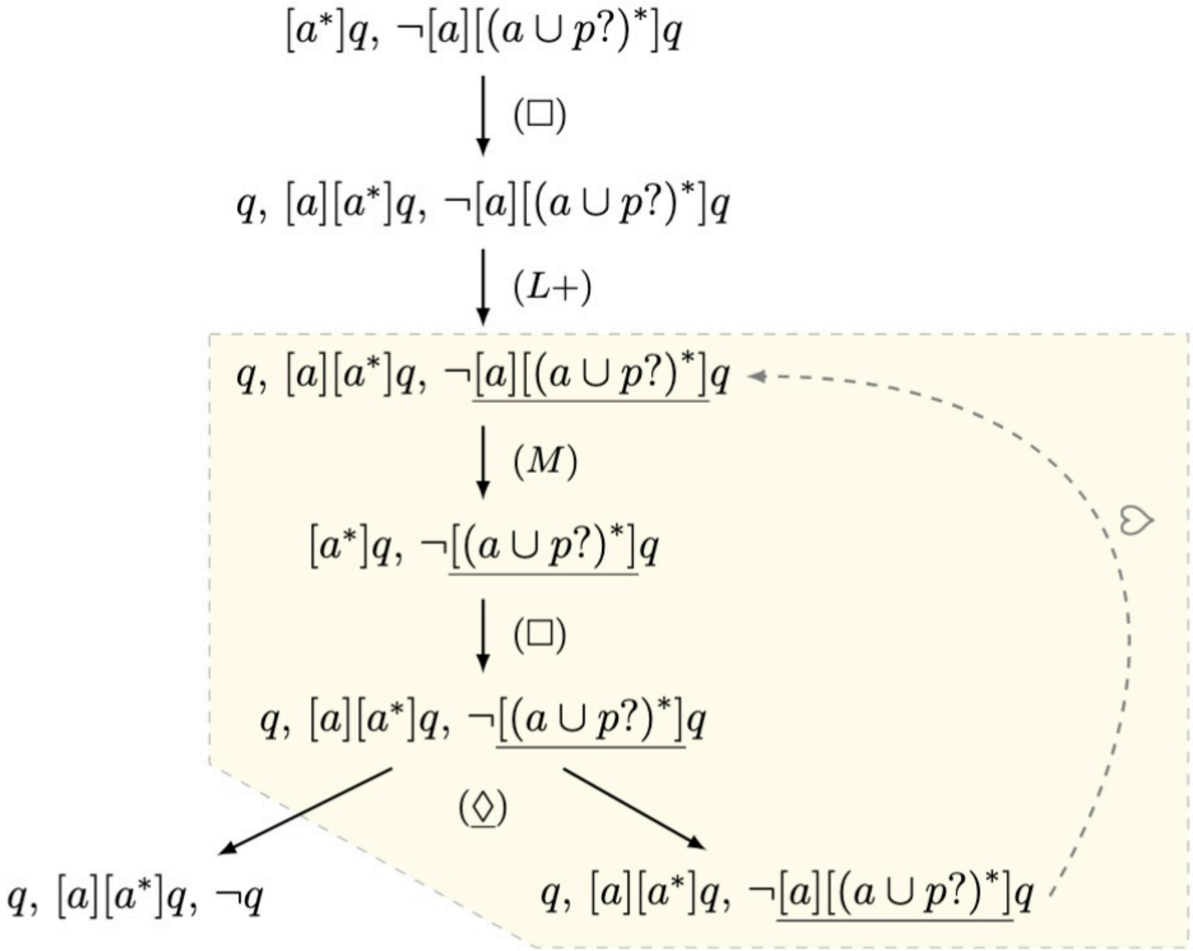
$$(\underline{\Diamond})\ \frac{\Delta, \neg\underline{[\alpha]}\xi}{\left\{\Delta, \Gamma \mid \Gamma \in \underline{\mathsf{unfold}}_{\Diamond}(\alpha, \xi)\right\}} \quad \alpha \text{ non-atomic}$$

Dealing with unguarded formulas

$$(L+)\ \frac{\Delta, \neg[a][\alpha_1]\ldots[\alpha_n]\varphi}{\Delta, \neg\underline{[a][\alpha_1]\ldots[\alpha_n]}\varphi} \qquad (L-)\ \frac{\Delta, \neg\underline{[\alpha_1]\ldots[\alpha_n]}\varphi}{\Delta, \neg[\alpha_1]\ldots[\alpha_n]\varphi}$$

$$(M)\ \frac{\Delta, \neg\underline{[a]}\xi}{\Delta_a, \neg\xi} \quad \Delta \text{ basic}$$

A PDL Tableau



Contributions

- A sound and complete Cyclic Tableau System for PDL with:
  - Unfold rules to deal with unguarded formulas
  - A loading mechanism to detect successful repeats
- A sound and complete Split Cyclic Tableau System for PDL
- A twist in Cyclic Maehara to be able to solve all the equations that arise in the construction of the interpolants.

  Quasitableaux!
- A proof that:

  Propositional Dynamic Logic has Craig Interpolation

Questions?

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Putting it all together