Uniform Interpolation, Nested Sequents, and Bisimulation Quantification

Iris van der Giessen (University of Birmingham)
Raheleh Jalali (Utrecht University)
Roman Kuznets (TUWien)

LLAMA Seminar – Amsterdam – 16 November 2022

THE JOURNAL OF SYMBOLIC LOGIC
Volume 57, Number 1, March 1992

ON AN INTERPRETATION OF SECOND ORDER QUANTIFICATION IN FIRST ORDER INTUITIONISTIC PROPOSITIONAL LOGIC

ANDREW M. PITTS

Abstract. We prove the following surprising property of Heyting's intuitionistic propositional calculus, IpC. Consider the collection of formulas, ϕ, built up from propositional variables $(p, q, r, \ldots)$ and falsity $(\perp)$ using conjunction $(\wedge)$, disjunction $(\vee)$ and implication $(\rightarrow)$. Write $\vdash \phi$ to indicate that such a formula is intuitionistically valid. We show that for each variable p and formula ϕ there exists a formula $A_p\phi$ (effectively computable from ϕ), containing only variables not equal to p which occur in ϕ, and such that for all formulas ψ not involving p, $\vdash \psi \rightarrow A_p\phi$ if and only if $\vdash \psi \rightarrow \phi$. Consequently quantification over propositional variables can be modelled in IpC, and there is an interpretation of the second order propositional calculus, IpC^2, in IpC which restricts to the identity on first order propositions.

An immediate corollary is the strengthening of the usual interpolation theorem for IpC to the statement that there are least and greatest interpolant formulas for any given pair of formulas. The result also has a number of interesting consequences for the algebraic counterpart of IpC, the theory of Heyting algebras. In particular we show that a model of IpC^2 can be constructed whose algebra of truth-values is equal to any given Heyting algebra.

Propositional quantification

- Modal language (CNF):

$$\varphi ::= p \mid \overline{p} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi.$$

  Implication is defined by DeMorgan's laws:

$$\varphi \to \psi := \overline{\varphi} \vee \psi.$$

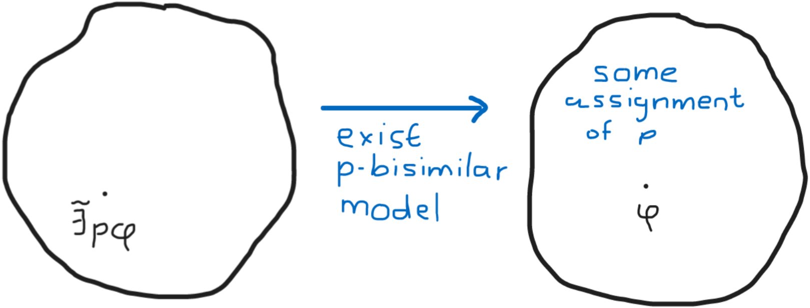
- What happens if we add quantifiers $\forall$ and $\exists$ that range over propositional variables?
- When are propositional quantifiers $\forall$ and $\exists$ definable within the modal language?

Bisimulation quantification

On the semantic side:

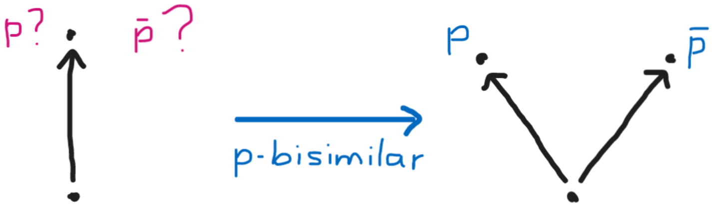
- In case the propositional quantifiers are definable within the propositional modal logic we have to take into account:

  Bisimilar models $\Rightarrow$ Modal equivalence



Bisimulation quantification - Example

- Consider $\tilde{\exists} p(\Diamond p \wedge \Diamond \overline{p})$.



Uniform interpolation

- Given modal formula $\varphi(p)$, what are the logical consequences of φ that do not contain p?

Def: A modal logic L has uniform interpolation if for all formulas φ and atoms p there are interpolants $\forall p\varphi$ and $\exists p\varphi$ such that

1. $\vdash_{\mathsf{L}} \forall p\varphi \to \varphi$ and $\vdash_{\mathsf{L}} \varphi \to \exists p\varphi$,
2. for all ψ not containing p we have

$$\vdash_{\mathsf{L}} \psi \to \varphi \;\Rightarrow \vdash_{\mathsf{L}} \psi \to \forall p\varphi,$$
$$\vdash_{\mathsf{L}} \varphi \to \psi \;\Rightarrow \vdash_{\mathsf{L}} \exists p\varphi \to \psi,$$

3. $Var(\forall p\varphi), Var(\exists p\varphi) \subseteq Var(\varphi) \setminus \{p\}$.

Lem. Uniform interpolation $\Rightarrow$ Craig interpolation

Uniform interpolation

- Given modal formula $\varphi(p)$, what are the logical consequences of φ that do not contain p?

Def: A modal logic L has uniform interpolation if for all formulas φ and atoms p there exist interpolant $\forall p \varphi$ such that

1. $\vdash_{\mathsf{L}} \forall p \varphi \to \varphi$
2. for all ψ not containing p we have

$$\vdash_{\mathsf{L}} \psi \to \varphi \; \Rightarrow \vdash_{\mathsf{L}} \psi \to \forall p \varphi,$$

3. $Var(\forall p \varphi), \qquad \subseteq Var(\varphi) \setminus \{p\}$.

Lem. Uniform interpolation $\Rightarrow$ Craig interpolation

Bisimulation quantification $\Rightarrow$ Uniform interpolation

Thm. If bisimulation quantifiers are definable in a class of models that is sound and complete w.r.t. logic L, then L has the uniform interpolation property.

Proof idea: Define $\forall p\varphi := \tilde{\forall} p\varphi$.

To prove:
For all formulas φ and atoms p there exist formula $\forall p\varphi$ s.t.

1. $\vdash_{\mathsf{L}} \forall p\varphi \to \varphi$,
2. for all ψ not containing p we have

$$\vdash_{\mathsf{L}} \psi \to \varphi \ \Rightarrow \vdash_{\mathsf{L}} \psi \to \forall p\varphi$$

Bisimulation quantification $\Rightarrow$ Uniform interpolation

Thm. If bisimulation quantifiers are definable in a class of models that is sound and complete w.r.t. logic L, then L has the uniform interpolation property.

Proof idea: Define $\forall p\varphi := \tilde{\forall} p\varphi$.

To prove:
For all formulas φ and atoms p there exist formula $\forall p\varphi$ s.t.

1. for all models $\mathcal{M}$ and worlds w, $\mathcal{M}, w \models \forall p\varphi$ implies $\mathcal{M}, w \models \varphi$,
2. for all ψ not containing p we have

$$\vdash_{\mathsf{L}} \psi \to \varphi \ \Rightarrow \vdash_{\mathsf{L}} \psi \to \forall p\varphi$$

Bisimulation quantification $\Rightarrow$ Uniform interpolation

Thm. If bisimulation quantifiers are definable in a class of models that is sound and complete w.r.t. logic L, then L has the uniform interpolation property.

Proof idea: Define $\forall p\varphi := \tilde{\forall} p\varphi$.

To prove:
For all formulas φ and atoms p there exist formula $\forall p\varphi$ s.t.

1. for all models $\mathcal{M}$ and worlds w,
   $\mathcal{M}, w \models \forall p\varphi$ implies $\mathcal{M}, w \models \varphi$,
2. for all ψ not containing p we have

$$\nvdash_{\mathsf{L}} \psi \to \forall p\varphi \;\Rightarrow \nvdash_{\mathsf{L}} \psi \to \varphi$$

Bisimulation quantification $\Rightarrow$ Uniform interpolation

Thm. If bisimulation quantifiers are definable in a class of models that is sound and complete w.r.t. logic L, then L has the uniform interpolation property.

Proof idea: Define $\forall p\varphi := \tilde{\forall} p\varphi$.

To prove:
For all formulas φ and atoms p there exist formula $\forall \boldsymbol{p}\varphi$ s.t.

1. for all models $\mathcal{M}$ and worlds w,
   $\mathcal{M}, w \models \forall \boldsymbol{p}\varphi$ implies $\mathcal{M}, w \models \varphi$,
2. for all ψ not containing p we have that

   there exists M, w s.t. $M, w \models \psi$ and $M, w \not\models \forall \boldsymbol{p}\varphi$

   implies

   there exists M', w' s.t. $M', w' \models \psi$ and $M', w' \not\models \varphi$

Bisimulation quantification $\Rightarrow$ Uniform interpolation

Thm. If bisimulation quantifiers are definable in a class of models that is sound and complete w.r.t. logic L, then L has the uniform interpolation property.

Proof idea: Define $\forall p\varphi := \tilde{\forall} p\varphi$.

To prove:
For all formulas φ and atoms p there exist formula $\forall p\varphi$ s.t.

1. for all models $\mathcal{M}$ and worlds w, $\mathcal{M}, w \models \forall p\varphi$ implies $\mathcal{M}, w \models \varphi$,
2. for all M, w s.t.

$$M, w \not\models \forall p\varphi$$

there is a p-bisimilar model M', w' s.t.

$$M', w' \not\models \varphi.$$

Examples

- CPC and S5
- IPC (Pitts ‘92)
- K (Ghilardi - Zawadowski ‘95) (Visser ‘96), GL (Shavrukov ‘94), T (Bílková ‘06)
- There are seven intermediate logics with uniform interpolation: $\mathrm{IPC}, \mathrm{Sm}, \mathrm{GSc}, \mathrm{LC}, \mathrm{KC}, \mathrm{Bd}_2, \mathrm{CPC}$ (Maksimova ‘77, Ghilardi - Zawadowski ‘02)
- iK and iKD (Iemhoff ‘19)

No uniform interpolation:

- S4 (Ghilardi - Zawadowski ‘95) and K4 (Bílková ‘06)
- iS4 and iK4 (vdG. ‘22)

Uniform interpolation via sequents

- Sequent $\Gamma \Rightarrow \Delta$, with formula interpretation $\bigwedge \Gamma \to \bigvee \Delta$.
- Proof-theoretic Maehara method for Craig interpolation.
- Proof-theoretic method for uniform interpolation (Pitts '92).

Craig interpolation	**uniform interpolation**
sequent-style definition of interpolation	sequent-style definition of uniform interpolation
cut-free calculus	terminating cut-free calculus
inductive definition using a proof	inductive definition using a proof search

Uniform interpolation via nested sequents

- A nested sequent is a tree of sequents
- Joint work with Roman Kuznets and Raheleh Jalali ('21)

UIP via sequents	UIP via nested sequents
sequent-style definition of uniform interpolation	nested sequent-style definition of uniform interpolation
terminating cut-free sequent calculus	terminating cut-free nested calculus (Brünnler '09)
inductive definition using a proof search	inductive definition using a proof search
syntactic proof of correctness	semantic proof of correctness with bisimulation quantifiers

Nested sequents

Def. Nested sequents Γ are recursively defined as follows:

$$\varphi_1, \ldots, \varphi_n, \left[\Gamma_1\right], \ldots, \left[\Gamma_m\right]$$

where $\varphi_1, \ldots, \varphi_n$ are modal formulas for $n \geq 0$ and $\Gamma_1, \ldots, \Gamma_m$ are nested sequents for $m \geq 0$.

- The formula interpretation ι is defined recursively by

$$\varphi_1 \vee \cdots \vee \varphi_n \vee \Box\iota(\Gamma_1) \vee \cdots \vee \Box\iota(\Gamma_m).$$

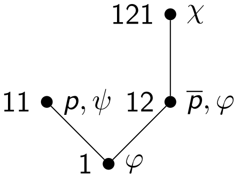
Nested sequents as trees

- Consider as an example

$$\Gamma = \varphi, [p, \psi], \big[\overline{p}, \varphi, [\chi]\big].$$

- We can label the nested sequent as follows:
  $l(\Gamma) = \varphi, [p, \psi]_{11}, \big[\overline{p}, \varphi, [\chi]_{121}\big]_{12}$ with set of labels
  $\mathcal{L}(\Gamma) = \{1, 11, 12, 121\}$.

- This can be pictured as a tree:



Terminating nested sequent calculi

- Terminating nested sequent calculi for logics K, T, and D developed by Brünnler ('09).

$$\mathsf{id_P}\,\frac{}{\Gamma\{p,\overline{p}\}} \qquad \mathsf{id}_\top\,\frac{}{\Gamma\{\top\}} \qquad \vee\,\frac{\Gamma\{\varphi\vee\psi,\varphi,\psi\}}{\Gamma\{\varphi\vee\psi\}}$$

$$\wedge\,\frac{\Gamma\{\varphi\wedge\psi,\varphi\} \quad \Gamma\{\varphi\wedge\psi,\psi\}}{\Gamma\{\varphi\wedge\psi\}} \qquad \Box\,\frac{\Gamma\{\Box\varphi,[\varphi]\}}{\Gamma\{\Box\varphi\}}$$

$$\mathsf{k}\,\frac{\Gamma\{\Diamond\varphi,[\Delta,\varphi]\}}{\Gamma\{\Diamond\varphi,[\Delta]\}} \qquad \mathsf{d}\,\frac{\Gamma\{\Diamond\varphi,[\varphi]\}}{\Gamma\{\Diamond\varphi\}} \qquad \mathsf{t}\,\frac{\Gamma\{\Diamond\varphi,\varphi\}}{\Gamma\{\Diamond\varphi\}}$$

Bisimulation nested sequent uniform interpolation

Def: Logic L has bisimulation nested sequent uniform interpolation property if for each nested sequent Γ and atom p there exists multiformula interpolant $A_p(\Gamma)$ such that for all models $\mathcal{M}$ and multiworld interpretations $\mathcal{I}$:

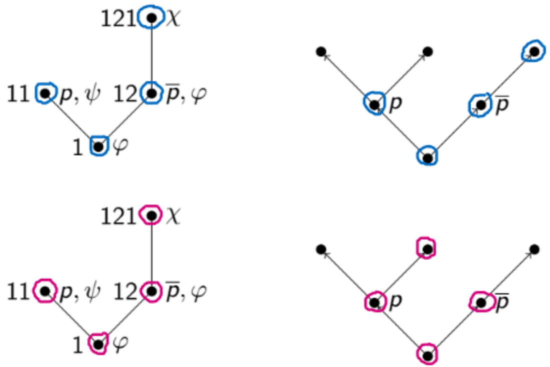
1. if $\mathcal{M}, \mathcal{I} \models A_p(\Gamma)$, then $\mathcal{M}, \mathcal{I} \models \Gamma$,
2. if $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$, then there is a p-bisimilar model $\mathcal{M}'$ with multiworld interpretation $\mathcal{I}'$ such that $\mathcal{M}', \mathcal{I}' \not\models \Gamma$,
3. variable condition and label condition.

Thm: K, T, and D have the bisimulation nested sequent uniform interpolation property.

Semantics for nested sequents

Def: A multiworld interpretation $\mathcal{I}$ of a nested sequent Γ into model $\mathcal{M} = (W, R, V)$ is a function from labels in Γ to worlds in $\mathcal{M}$ such that $\mathcal{I}(\sigma) R \mathcal{I}(\sigma * n)$ whenever $\{\sigma, \sigma * n\} \subseteq \mathcal{L}(\Phi)$.

Def: Define $\mathcal{M}, \mathcal{I} \models \Gamma$ iff $\mathcal{M}, \mathcal{I}(\sigma) \models \varphi$ for some $\sigma : \varphi \in \Gamma$.



Multiformulas

Def: Multiformulas:

$$\mho ::= \sigma : \varphi \mid \mho \ovee \mho \mid \mho \owedge \mho$$

Def: Define $\mathcal{M}, \mathcal{I} \models \mho$ recursively:

$$\begin{array}{lll}
\mathcal{M}, \mathcal{I} \models \sigma : \varphi & \text{iff} & \mathcal{M}, \mathcal{I}(\sigma) \models \varphi, \\
\mathcal{M}, \mathcal{I} \models \mho_1 \owedge \mho_2 & \text{iff} & \mathcal{M}, \mathcal{I} \models \mho_1 \text{ and } \mathcal{M}, \mathcal{I} \models \mho_2, \\
\mathcal{M}, \mathcal{I} \models \mho_1 \ovee \mho_2 & \text{iff} & \mathcal{M}, \mathcal{I} \models \mho_1 \text{ or } \mathcal{M}, \mathcal{I} \models \mho_2.
\end{array}$$

Ex: $\sigma : \varphi \owedge \sigma : \psi \equiv \sigma : (\varphi \wedge \psi)$.

Bisimulation nested sequent uniform interpolation

Def: Logic L has bisimulation nested sequent uniform interpolation property if for each nested sequent Γ and atom p there exists multiformula interpolant $A_p(\Gamma)$ such that for all models $\mathcal{M}$ and multiworld interpretations $\mathcal{I}$:

1. if $\mathcal{M}, \mathcal{I} \models A_p(\Gamma)$, then $\mathcal{M}, \mathcal{I} \models \Gamma$,
2. if $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$, then there is a p-bisimilar model $\mathcal{M}'$ with multiformula interpretation $\mathcal{I}'$ such that $\mathcal{M}', \mathcal{I}' \not\models \Gamma$,
3. variable condition and label condition.

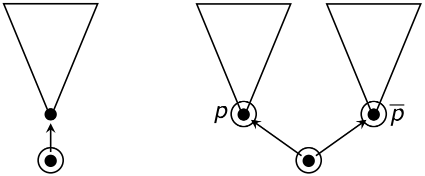
Thm: K, T, and D have the bisimulation nested sequent uniform interpolation property.

Example $A_p(\Box p, \Box \overline{p})$

- Proof search: $$\dfrac{\dfrac{\Box p, \Box \overline{p}, [p]_{11}, [\overline{p}]_{12}}{\Box p, \Box \overline{p}, [p]_{11}}}{\Box p, \Box \overline{p}} \qquad \begin{array}{c} 1 : \bot \owedge 11 : \bot \owedge 12 : \bot \\ 1 : \Box\bot \owedge 11 : \bot \\ 1 : \Box\bot \owedge 1 : \Box\bot \end{array}$$

- Recursively define the multiformula interpolant: $1 : \Box\bot$.

- Prove correctness:
  1. if $\mathcal{M}, \mathcal{I} \models 1 : \Box\bot$, then $\mathcal{M}, \mathcal{I} \models \Box p, \Box \overline{p}$.
  2. if $\mathcal{M}, \mathcal{I} \not\models 1 : \Box\bot$, then there is a p-bisimilar model $\mathcal{M}'$ with multiformula interpretation $\mathcal{I}'$ such that $\mathcal{M}', \mathcal{I}' \not\models \Box p, \Box \overline{p}$.



To conclude

Done:

- Proof for K, T and D (vdG., Jalali, Kuznets ‘21), (vdG. ‘22, Chp. 4)
- Similar strategy for hypersequents for S5 (vdG., Jalali, Kuznets, WoLLIC)(vdG. ‘22, Chp. 4)

Working on:

- Proof for K5 with grafted hypersequents.

Future:

- Modular method / general properties of the method.
- Prove uniform interpolation for iSL and iGL.
  - Terminating calculi in (vdG. ‘22, Chp. 3, j.w.w. Iemhoff) Please see pp. 78–81.

Related work:

- A note on uniform interpolation proofs in modal deep inference calculi (Bílková ‘11).