A uniform interpolant calculator with Coq

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Uniform interpolation in many forms

Knowledge representation

Automated deduction

Proof Theory

Algebra

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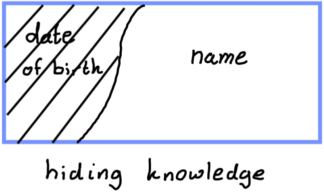

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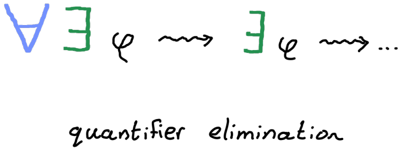

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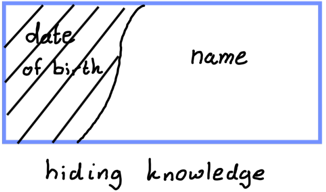

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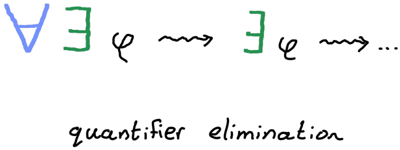

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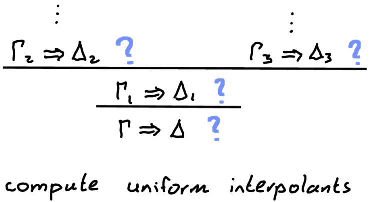

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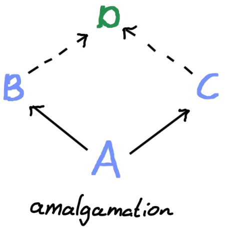

Our work in short

What?
Mechanised uniform interpolation (for modal logics) in Coq ( )

Why?
Our formalisation provides the following benefits:

Our work in short

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- Proof-theoretic construction of uniform interpolants in Coq
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Our formalisation provides the following benefits:

- Uniform interpolants for new logics $\rightarrow$ iSL
- Fix bugs in paper-and-pen proofs $\rightarrow$ GL
- Use calculator to derive theoretical results [Kocsis '23]

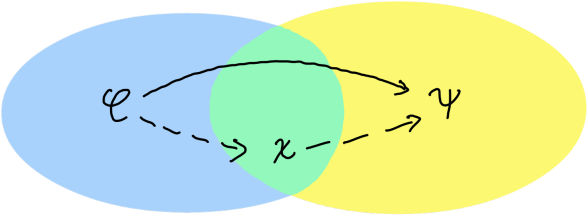
Craig interpolation

Language: $\varphi ::= p \in Var \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi$

Definition

Let L be a (modal) logic. L has the *Craig interpolation property* if for any $\vdash_{\mathsf{L}} \varphi \to \psi$ there exists formula χ such that both:

1. $Var(\chi) \subseteq Var(\varphi) \cap Var(\psi)$
2. $\vdash_{\mathsf{L}} \varphi \to \chi$ and $\vdash_{\mathsf{L}} \chi \to \psi$



Uniform interpolation

Slogan

A Craig interpolant for $\varphi \to \psi$ that only depends on φ is *uniform*.

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Definition

Logic L has the *(right) uniform interpolation property* if for any formula φ and atom p there exists formula χ such that:

1. $Var(\chi) \subseteq Var(\varphi) \setminus \{p\}$
2. For all p-free formulas ψ:

$$\text{if } \vdash_{\mathsf{L}} \varphi \to \psi, \text{ then } \vdash_{\mathsf{L}} \varphi \to \chi \text{ and } \vdash_{\mathsf{L}} \chi \to \psi$$

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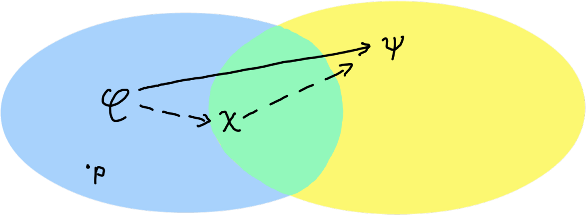

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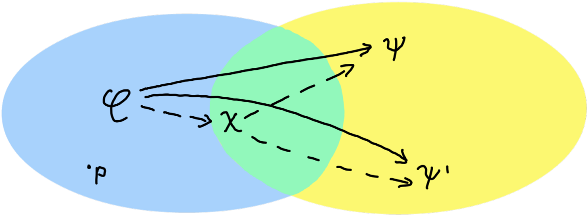

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Logic L has the *(right) uniform interpolation property* if for any formula φ and atom p there exists formula $\exists p\varphi$ such that:

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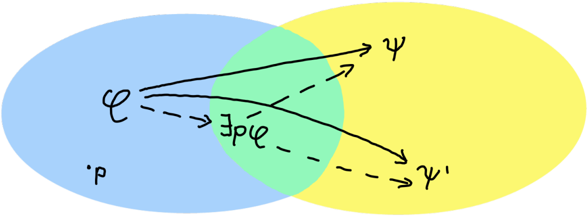

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Logic L has the *(left) uniform interpolation property* if for any formula φ and atom *p* there exists formula $\forall p \varphi$ such that:

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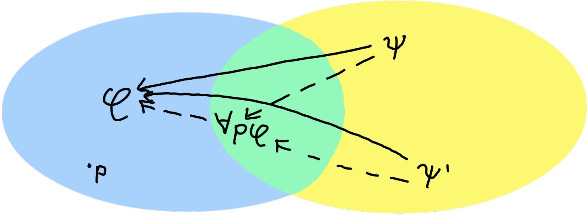

Example: classical logic

Uniform interpolants form strongest ($\exists$) and weakest ($\forall$) interpolants.

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$$\varphi = p \wedge q \wedge r$$
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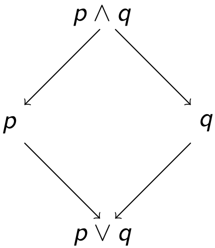

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$$\exists r \varphi = (p \wedge q \wedge \top) \vee (p \wedge q \wedge \bot) = p \wedge q$$

$$p \qquad\qquad q$$

$$\forall s \psi = (p \vee q \vee \top) \wedge (p \vee q \vee \bot) = p \vee q$$

Difficulties in constructing uniform interpolants

Given a logic with Craig interpolation, we define

$$\exists p.\varphi(p,\overline{q}) := \bigwedge\{\psi(\overline{q}) \ \mid\ \vdash \varphi \to \psi\}$$
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Idea:
Replace it by

$$E_p(\varphi) := \bigwedge \mathcal{E}_p(\varphi) \quad \text{and} \quad A_p(\varphi) := \bigvee \mathcal{A}_p(\varphi)$$

where $\mathcal{E}_p(\varphi)$ is a finite basis for the set of consequences of φ (dually for $\mathcal{A}_p(\varphi)$)

Pitts's solution: a finite base via proof theory

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Examples

- IPC [Pitts '92]
- Classical modal logics K, T, GL [Bílková '06]
- Intuitionistic modal logics iK, iKD [Iemhoff '19]
- Our paper: K, GL and iSL

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Disclaimer

This is an easy example. In general, the uniform interpolant is computed via recursive calls on *multiple* proof searches.

Terminating sequent calculi

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- Replace non-terminating rule

$$\frac{\Gamma, \varphi \to \psi \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \gamma}{\Gamma, \varphi \to \psi \Rightarrow \gamma}\ \mathrm{L}{\to}$$

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- Strong termination in multiset ordering
  → We use a *numerical* ordering assigning weights to sequents

Terminating sequent calculi

For classical Gödel-Löb logic GL: [Bílková 06]

- Modal rule

$$\frac{\Box\Gamma, \Gamma, \Box\varphi \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \text{ GL-rule}$$

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The quest for a suitable calculus for iSL:

- Terminating with local loop-check [vdG & Iemhoff '20] ✗
- Strongly terminating [Shillito, vdG, Goré & Iemhoff '23] ✓
- Terminating and subformula property [Fiorentini & Ferrari '24] ?

Construction for iSL

	Γ matches	$\mathcal{E}_p(\Gamma)$ contains
($\mathsf{E}_p^{\mathsf{IL}}0$)	$\Gamma', \bot$	$\bot$
($\mathsf{E}_p^{\mathsf{IL}}1$)	Γ', q	$\mathsf{E}_p(\Gamma') \wedge q$
($\mathsf{E}_p^{\mathsf{IL}}2$)	$\Gamma', \psi_1 \wedge \psi_2$	$\mathsf{E}_p(\Gamma', \psi_1, \psi_2)$
($\mathsf{E}_p^{\mathsf{IL}}3$)	$\Gamma', \psi_1 \vee \psi_2$	$\mathsf{E}_p(\Gamma', \psi_1) \vee \mathsf{E}_p(\Gamma', \psi_2)$
($\mathsf{E}_p^{\mathsf{IL}}4$)	$\Gamma', (q \to \psi)$	$q \to \mathsf{E}_p(\Gamma', \psi)$
($\mathsf{E}_p^{\mathsf{IL}}5$)	$\Gamma', p, (p \to \psi)$	$\mathsf{E}_p(\Gamma', p, \psi)$
($\mathsf{E}_p^{\mathsf{IL}}6$)	$\Gamma', (\delta_1 \wedge \delta_2) \to \delta_3)$	$\mathsf{E}_p(\Gamma', (\delta_1 \to (\delta_2 \to \delta_3)))$
($\mathsf{E}_p^{\mathsf{IL}}7$)	$\Gamma', (\delta_1 \vee \delta_2) \to \delta_3)$	$\mathsf{E}_p(\Gamma', (\delta_1 \to \delta_3), (\delta_2 \to \delta_3)))$
($\mathsf{E}_p^{\mathsf{IL}}8$)	$\Gamma', (\delta_1 \to \delta_2) \to \delta_3)$	$[\mathsf{E}_p(\Gamma', (\delta_2 \to \delta_3)) \to \mathsf{A}_p(\Gamma', (\delta_2 \to \delta_3) \Rightarrow \delta_1 \to \delta_2)]$ $\to \mathsf{E}_p(\Gamma', \delta_3)$
($\mathsf{E}_p^{\mathsf{iSL}}9$)	$\Gamma', \Box\delta$	$\Box\mathsf{E}_p(\otimes\Gamma', \delta)$
($\mathsf{E}_p^{\mathsf{iSL}}10$)	$\Gamma', (\Box\delta_1 \to \delta_2)$	$\Box[\mathsf{E}_p(\otimes\Gamma', \delta_2, \Box\delta_1) \to \mathsf{A}_p(\otimes\Gamma', \delta_2, \Box\delta_1 \Rightarrow \delta_1)]$ $\to \mathsf{E}_p(\Gamma', \delta_2)$

	s matches	$\mathcal{A}_p(s)$ contains
($\mathsf{A}_p^{\mathsf{IL}}1$)	$\Gamma, q \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma \Rightarrow \varphi)$
($\mathsf{A}_p^{\mathsf{IL}}2$)	$\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma, \psi_1, \psi_2 \Rightarrow \varphi)$
($\mathsf{A}_p^{\mathsf{IL}}3$)	$\Gamma, \psi_1 \vee \psi_2 \Rightarrow \varphi$	$[\mathsf{E}_p(\Gamma, \psi_1) \to \mathsf{A}_p(\Gamma, \psi_1 \Rightarrow \varphi)] \wedge$ $[\mathsf{E}_p(\Gamma, \psi_2) \to \mathsf{A}_p(\Gamma, \psi_2 \to \varphi)]$
($\mathsf{A}_p^{\mathsf{IL}}4$)	$\Gamma, (q \to \psi) \Rightarrow \varphi$	$q \wedge \mathsf{A}_p(\Gamma, \psi \Rightarrow \varphi)$
($\mathsf{A}_p^{\mathsf{IL}}5$)	$\Gamma, p, (p \to \psi) \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma, \psi \Rightarrow \varphi)$

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Problem for GL

Similar sequents may have different proof search trees. E.g.,

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This results in different constructions of $\mathcal{A}_p$ circumventing the fixpoint argument used in [Bílková '06] for GL.

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Our solution

Enforce equality of sequents in the computation by using contraction

Our formalisation in Coq

Why?

- Better understand a 30-year old proof [Férée & v. Gool '23]
- Use the implementation to derive theoretical results
  - Non-definability of connectives [Kocsis '23]
- 'Easily' derive uniform interpolants for new logics → iSL
- Fix bugs in paper-and-pen proofs → GL

Challenges of our formalisation

- Prove termination of the algorithm
- Formalise 'easy' admissibility and inversion of rules [Dyckhoff & Negri '20]
- Work in progress: output readable interpolants

It's time for a

Coq $\Rightarrow$ OCaml program $\Rightarrow$ Javascript program

Coq documentation: `https://hferee.github.io/UIML/toc.html`
Calculator: `https://hferee.github.io/UIML/demo.html`

Conclusion

This paper

- Uniform Interpolation Calculator for modal logics in Coq
- Fix bug for GL in [Bílková '06]
- First proof of uniform interpolation for iSL
  → semantic proof in [Litak & Visser '24]

Work in progress

- Simplify interpolants
- Connect proof-theoretic with semantic methods
- How to tackle difficult cases (like iGL)?

My plan for three years at UvA