Towards Constructive Soundness for Cyclic Heyting Arithmetic

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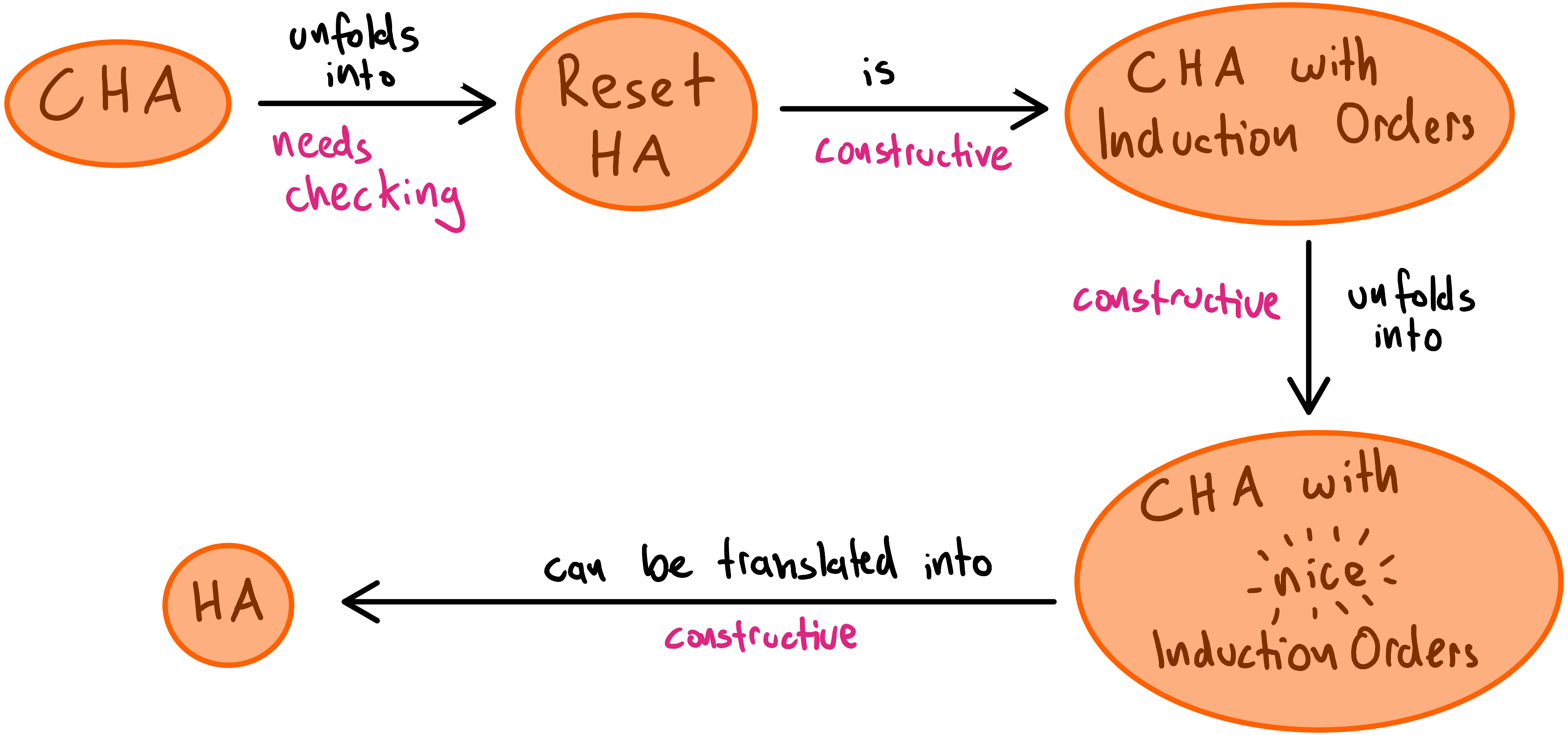
at the

LLAMA seminar on the 7th of September 2022

The Plan for Today



References

Cyclic HA: Equivalence of Intuitionistic Inductive Definitions and Intuitionistic Cyclic Proofs under Arithmetic Berardi, Tatsuta 2017

Reset Proofs: Tableau Systems for the Modal μ-Calculus Jungteerapanich 2010

A Proof System with Names for Modal Mu-Calculus Stirling 2013

Induction Orders & Translation Procedure: On the Structure of Inductive Reasoning: Circular and Tree-Shaped Proofs in the μCalculus Sprenger, Dam 2003

1. Heyting Arithmetic and Cyclic Heyting Arithmetic

Deduction Systems: IQ & HA

IQ

- intuitionistic FO sequent rules
- "usual" axioms for $S, +, \cdot$

HA

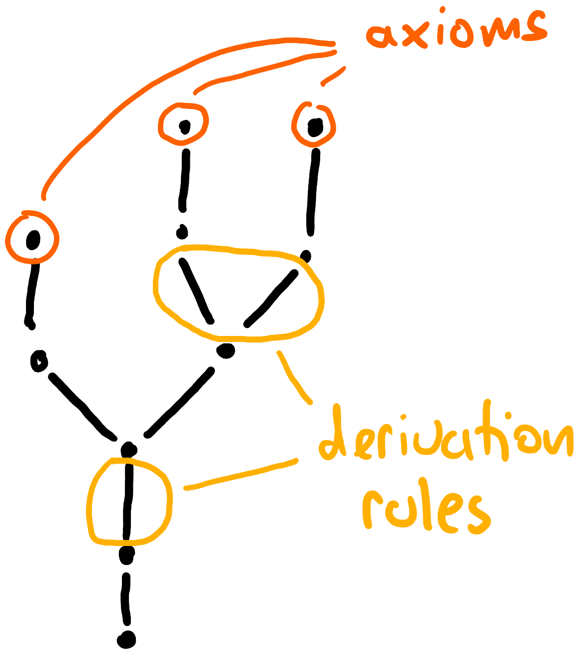
Induction scheme:

$$\varphi(0) \to (\forall x.\varphi(x) \to \varphi(Sx)) \to \forall x.\varphi(x)$$

Sequents: $\Gamma \Rightarrow \varphi$

Γ a finite set of FOA fmlas

Proofs:

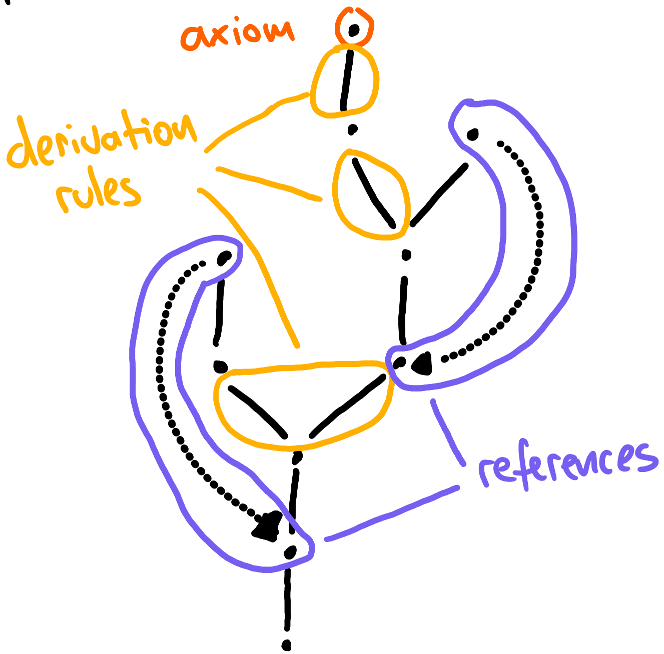


Deduction Systems: Cyclic HA

Rules + axioms:

$$\mathsf{IQ} + \quad \text{Case}\ \frac{\Gamma(0) \Rightarrow \varphi(0) \qquad \Gamma(Sx) \Rightarrow \varphi(Sx)}{\underbrace{\Gamma(x) \Rightarrow \varphi(x)}_{x \notin FV(\Gamma, \varphi)}}$$

Pre-proofs:



References:

$$\begin{array}{c} \Gamma \Rightarrow \varphi \\ \hline \vdots \ \pi \\ \hline \Gamma \Rightarrow \varphi \end{array}$$

Condition:
Must be the same sequent!

Cyclic HA: Example: Induction Scheme for $\varphi(x)$

$$
\begin{prooftree}
\AxiomC{$\varphi(0) \Rightarrow \varphi(0)$}
\RightLabel{Wk}
\UnaryInfC{$\varphi(0), (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(0)$}
\AxiomC{$\varphi(Sx) \Rightarrow \varphi(Sx)$}
\LeftLabel{Wk}
\UnaryInfC{$\varphi(0), (\forall x.\varphi(x) \to \varphi(Sx)), \varphi(Sx) \Rightarrow \varphi(Sx)$}
\AxiomC{$\boxed{\varphi(0), (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}$}
\LeftLabel{$\to$L}
\BinaryInfC{$\varphi(0), (\forall x.\varphi(x) \to \varphi(Sx)), \varphi(x) \to \varphi(Sx) \Rightarrow \varphi(Sx)$}
\RightLabel{$\forall$L}
\UnaryInfC{$\varphi(0), (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(Sx)$}
\RightLabel{Case}
\BinaryInfC{$\varphi(0), (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)$}
\RightLabel{$\to$R, $\forall$R}
\UnaryInfC{$\Rightarrow \varphi(0) \to (\forall x.\varphi(x) \to \varphi(Sx)) \to \forall x.\varphi(x)$}
\end{prooftree}
$$

Cyclic HA: A counterexample?

$$\mathrm{Cut}\;\dfrac{\mathrm{Wk}\;\dfrac{\boxed{\Rightarrow \bot}}{\bot \Rightarrow \bot} \qquad \boxed{\Rightarrow \bot}}{\Rightarrow \bot}$$

Cyclic HA: Proper Proofs

Global **T**race **C**ondition

Def: A CHA pre-proof is a proof if along every infinite path through it there exists a variable eventually present in all sequents along the path which is subject to the Case-rule infinitely often

Case! Case! ... ∞ often

$x \in FV(\Gamma_2, \varphi_2)$ $x \in FV(\Gamma_3, \varphi_4)$ ... $x \in FV(\Gamma_i, \varphi_i)$ ... forever

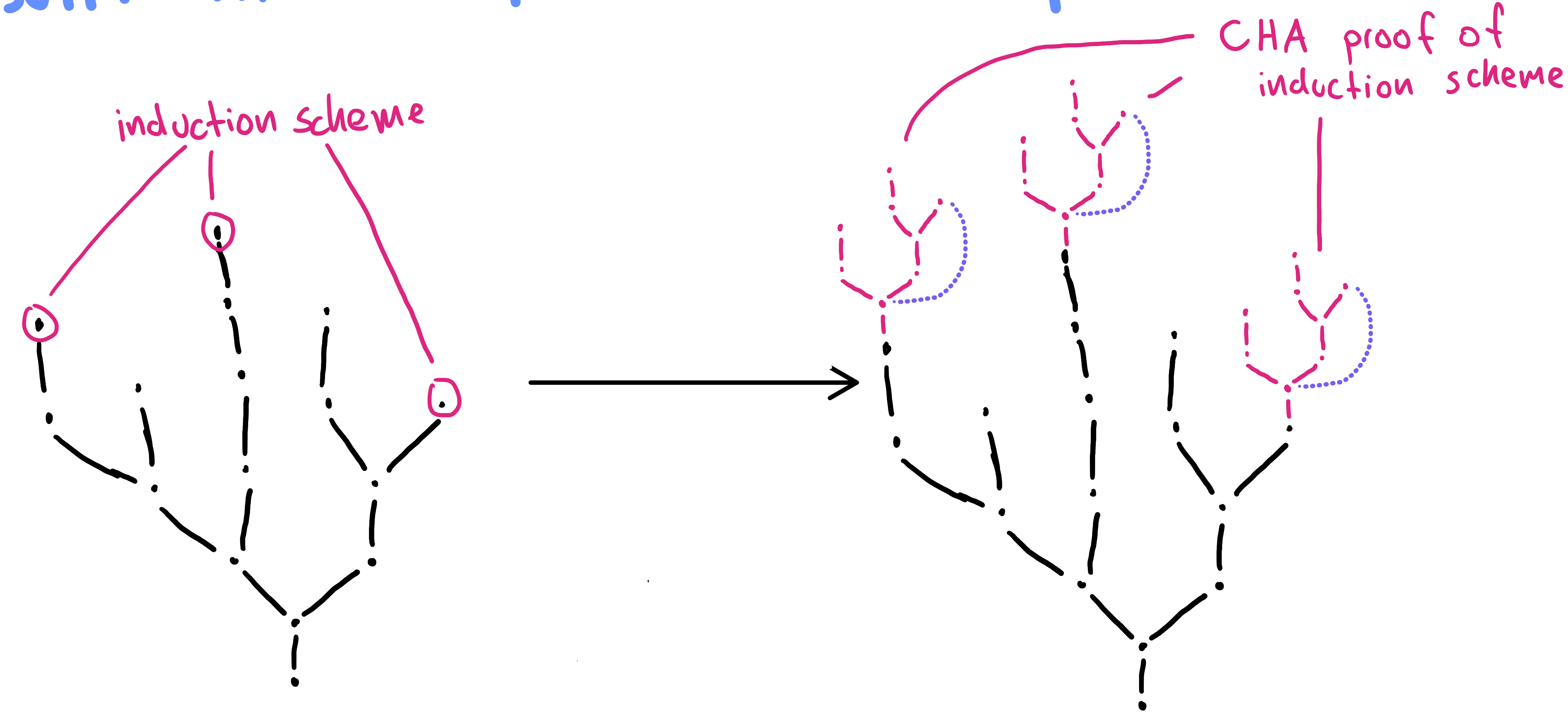
$\Gamma_0 \Rightarrow \varphi_0$ $\Gamma_1 \Rightarrow \varphi_1$ $\Gamma_2 \Rightarrow \varphi_2$ $\Gamma_3 \Rightarrow \varphi_3$ ... $\Gamma_i \Rightarrow \varphi_i$ ...

Cyclic HA: Example: Induction Scheme for $\varphi(x)$

<u>Def:</u> A CHA pre-proof is a proof if along every infinite path through it there exists a variable eventually present in all sequents along the path which is subject to the Case-rule infinitely often

$$
\dfrac{
\dfrac{
\dfrac{\varphi(0) \Rightarrow \varphi(0)}{\varphi(0),\ (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(0)}\ \mathrm{Wk}
\qquad
\dfrac{
\dfrac{
\mathrm{Wk}\ \dfrac{\varphi(Sx) \Rightarrow \varphi(Sx)}{\varphi(0),\ (\forall x.\varphi(x) \to \varphi(Sx)),\ \varphi(Sx) \Rightarrow \varphi(Sx)}
\qquad
\boxed{\varphi(0),\ (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}
}{\varphi(0),\ (\forall x.\varphi(x) \to \varphi(Sx)),\ \varphi(x) \to \varphi(Sx) \Rightarrow \varphi(Sx)}\ {\to}L
}{\varphi(0),\ (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(Sx)}\ \forall L
}{\varphi(0),\ (\forall x.\varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}\ \text{Case}
}{\Rightarrow \varphi(0) \to (\forall x.\varphi(x) \to \varphi(Sx)) \to \forall x.\varphi(x)}\ {\to}R, \forall R
$$

Result: $HA \vdash \Gamma \Rightarrow \varphi \longrightarrow CHA \vdash \Gamma \Rightarrow \varphi$



$$HA \vdash \Gamma \Rightarrow \varphi \longrightarrow CHA \vdash \Gamma \Rightarrow \varphi$$

Soundness of CHA, Classically

Thm: If $CHA \vdash \Gamma \Rightarrow \varphi$ then $\omega \models \bigwedge \Gamma \to \varphi$

Proof: Suppose $\rho \not\models \bigwedge \Gamma \Rightarrow \varphi$ for $\rho : \mathcal{V} \to \omega$.

Starting at the root of $CHA \vdash \Gamma \Rightarrow \varphi$ obtain an infinite path $(\Gamma_i \Rightarrow \varphi_i)_{i \in \omega}$ through the proof with assignments $\rho_i : \mathcal{V} \to \omega$ st. $\rho_i \not\models \bigwedge \Gamma_i \to \varphi_i$.

<u>Case $\forall R$:</u>

$$\frac{\Gamma_i \Rightarrow \varphi(x)}{\Gamma_i \Rightarrow \forall x. \varphi(x)} \quad \longrightarrow$$

There must exist n st. $\rho_i[x \mapsto n] \not\models \bigwedge \Gamma_i \to \varphi(x)$.

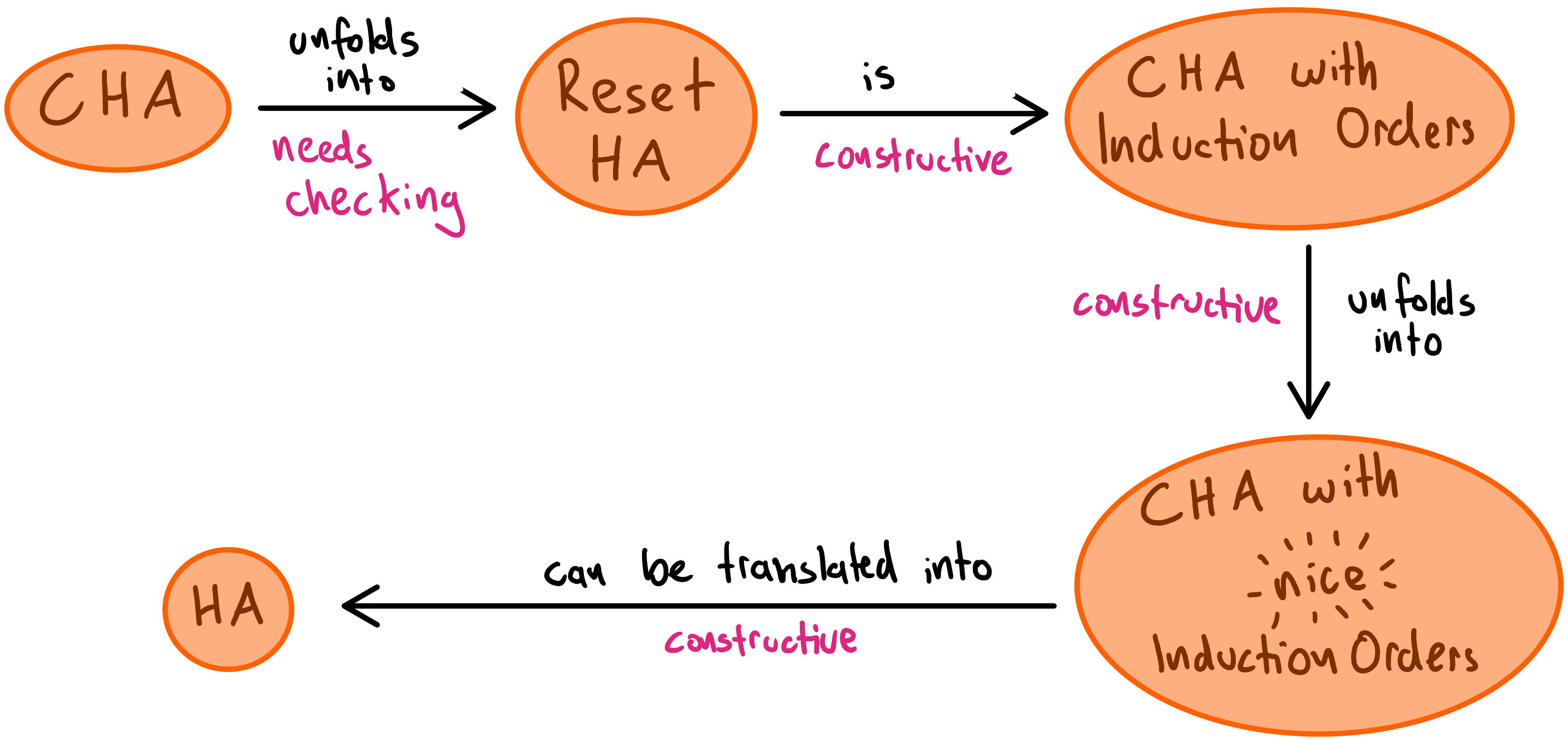
Pick $\Gamma_{i+1} := \Gamma_i$ $\quad \varphi_{i+1} := \varphi(x)$ $\quad \rho_{i+1} := \rho[x \mapsto n]$

$\vdots$

By the GTC there exists a variable x st $x \in FV(\Gamma_i, \varphi_i)$ for $i \geq N$.

Then $\rho_i(x) \geq \rho_{i+1}(x)$ with $\rho_i(x) > \rho_{i+1}(x)$ infinitely often ↯

The Plan for Today



2. Reset Heyting Arithmetic

Reset HA

Idea: Annotate pre-proofs with Safra automata runs

Sequents: $\underbrace{\Theta}_{\text{duplicate-free sequence of labels}} \; ; \; \underbrace{\sigma}_{\text{labelings of variables with subsequences of } \Theta} \; \vdash \; \underbrace{\Gamma \Rightarrow \varphi}_{\text{HA sequent}}$

Example: $\underbrace{abc}_{\Theta} \; ; \; \underbrace{x \mapsto a, \; y \mapsto bc}_{\sigma} \; \vdash \; \underbrace{x = 2 \Rightarrow y = 3}_{\Gamma \Rightarrow \varphi}$

Reset HA: Case rule

$$\underbrace{\Theta}_{\text{duplicate-free sequence of labels}} \; ; \; \underbrace{\sigma}_{\text{labelings of variables}} \vdash \underbrace{\Gamma \Rightarrow \varphi}_{\text{HA sequent}}$$

(where a does not occur in Θ)

$$\text{Case}\ \frac{\overline{\Theta}\,;\,\sigma \vdash \Gamma(0) \Rightarrow \varphi(0) \qquad \Theta_a\,;\,\sigma,\ x \mapsto u\,a \vdash \Gamma(Sx) \Rightarrow \varphi(Sx)}{\Theta\,;\,\sigma,\ x \mapsto u \vdash \Gamma(x) \Rightarrow \varphi(x)}$$

Reset HA: Reset Rule

$$\underbrace{\Theta}_{\text{duplicate-free sequence of labels}} \;;\; \underbrace{\sigma}_{\text{labelings of variables}} \;\vdash\; \underbrace{\Gamma \Rightarrow \varphi}_{\text{HA sequent}}$$

$$\text{Reset}_a \; \frac{\Theta\,;\,\sigma,\; x_1 \mapsto u_1 a,\; x_2 \mapsto u_2 a, \dots,\; x_n \mapsto u_n a \vdash \Gamma \Rightarrow \varphi}{\Theta\,;\,\sigma,\; x_1 \mapsto u_1 a u_1',\; x_2 \mapsto u_2 a u_2', \dots,\; x_n \mapsto u_n a u_n' \vdash \Gamma \Rightarrow \varphi}$$

no occurrences of a

may be empty

must not be empty

Effect: $x_i \mapsto u\,a\,u' \rightsquigarrow x_i \mapsto u\,a$

Reset HA: Other Rules

As usual + bookkeeping

$$\underbrace{\Theta}_{\text{duplicate-free sequence of labels}} \; ; \; \underbrace{\sigma}_{\text{labelings of variables}} \vdash \underbrace{\Gamma \Rightarrow \varphi}_{\text{HA sequent}}$$

Ensure $\mathrm{dom}(\bar\sigma) = FV(\Gamma, \varphi)$ and each label in $\bar\Theta$ is used by $\bar\sigma$

$$\frac{\bar\Theta, \bar\sigma \vdash \Gamma \Rightarrow \varphi}{\Theta, \sigma \vdash \Gamma \Rightarrow \varphi \vee \psi}\ \vee R_1$$

Add empty labeling for new variable

$$\frac{\Theta, \sigma, x \mapsto \varepsilon \vdash \Gamma \Rightarrow \varphi}{\Theta, \sigma \vdash \Gamma \Rightarrow \forall x. \varphi}\ \forall R$$

Reset HA: Proper Proofs

Reset Condition

A Reset HA preproof is a proof if for every cycle $(\Theta_i ; \sigma_i \vdash \Gamma_i \Rightarrow \varphi_i)_{i \leq n}$ there exists a label $a \in \Theta_i$ for all $i \leq n$ and an a-Reset takes place along the cycle

$$\begin{array}{cl}
\boxed{\Theta_a \, \Theta'_n ; \sigma_n \vdash \Gamma_n \Rightarrow \varphi_n} & \\ \hline
\vdots & \\ \hline
\Theta_a \, \Theta'_{i+1} ; \sigma_{i+1} \vdash \Gamma_{i+1} \Rightarrow \varphi_{i+1} & \text{Reset}_a \\ \hline
\Theta_a \, \Theta'_i ; \sigma_i \vdash \Gamma_i \Rightarrow \varphi_i & \\ \hline
\vdots & \\ \hline
\Theta_a \, \Theta'_2 ; \sigma_2 \vdash \Gamma_2 \Rightarrow \varphi_2 & \\ \hline
\Theta_a \, \Theta'_1 ; \sigma_1 \vdash \Gamma_1 \Rightarrow \varphi_1 &
\end{array}$$

Result: $\mathsf{CHA} \vdash \Gamma \Rightarrow \varphi \longrightarrow \mathsf{RHA} \vdash \Gamma \Rightarrow \varphi$

$$\boxed{\varphi(0), (\forall x. \varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}$$

Π (Contains Case)

$$\dfrac{\dfrac{\varphi(0), (\forall x. \varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}{\vdots}}{\Rightarrow \varphi(0) \to (\forall x. \varphi(x) \to \varphi(Sx)) \to \forall x. \varphi(x)}$$

$\rightsquigarrow$

$$\boxed{a;\ x \mapsto a \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)}$$

$$\dfrac{\boxed{a;\ x \mapsto a \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)}}{ab;\ x \mapsto ab \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)}$$

Reseta

Π

$$a;\ x \mapsto a \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)$$

Π

$$\dfrac{\dfrac{\varepsilon;\ x \mapsto \varepsilon \vdash \varphi(0), (\forall x. \varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}{\vdots}}{\varepsilon;\ x \mapsto \varepsilon \vdash \Rightarrow \varphi(0) \to (\forall x. \varphi(x) \to \varphi(Sx)) \to \forall x. \varphi(x)}$$

$$\mathsf{CHA} \vdash \varphi(0) \to (\forall x. \varphi(x) \to \varphi(Sx)) \to \forall x. \varphi(x) \rightsquigarrow \mathsf{RA} \vdash \varphi(0) \to (\forall x. \varphi(x) \to \varphi(Sx)) \to \forall x. \varphi(x)$$

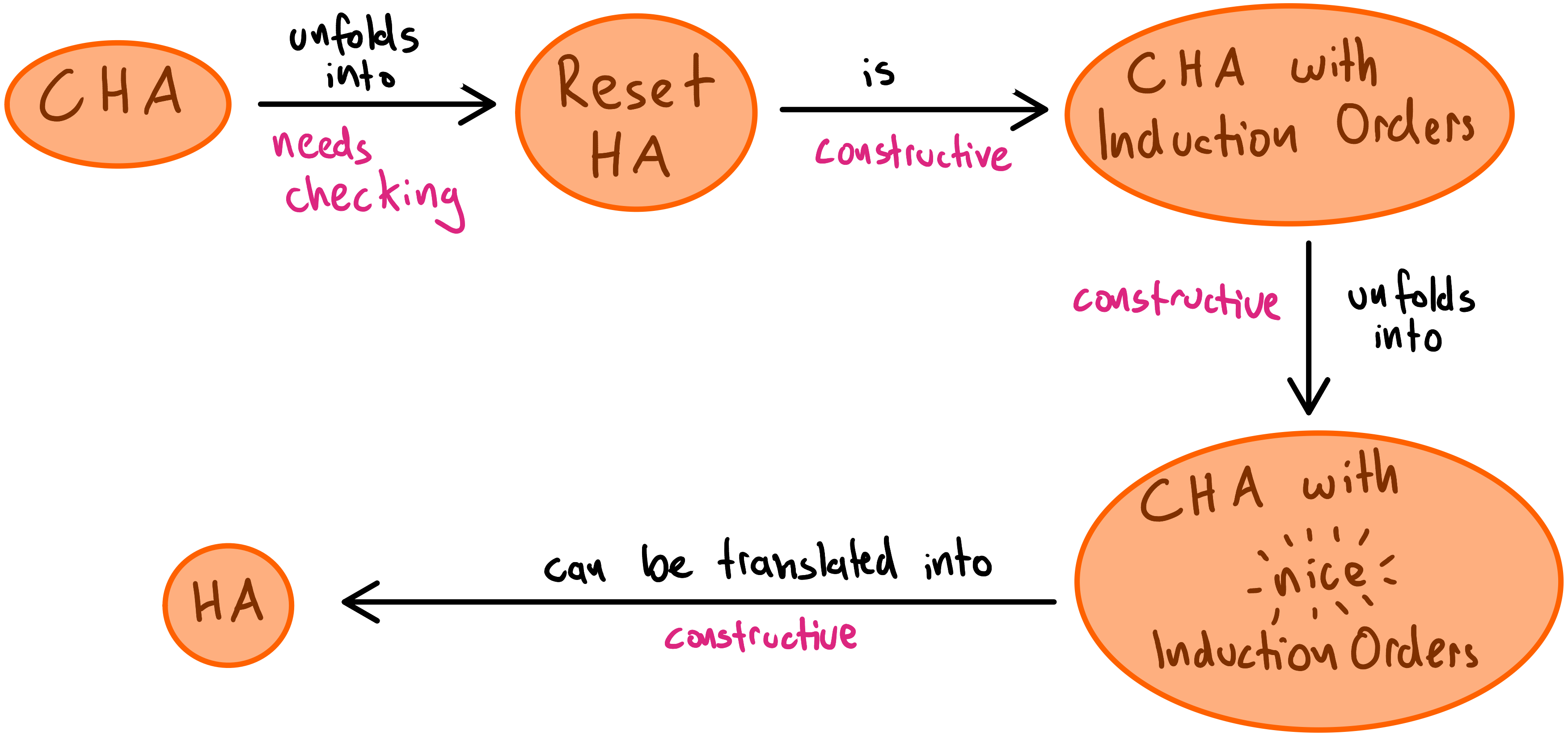
Result: $CHA \vdash \Gamma \Rightarrow \varphi \longrightarrow RHA \vdash \Gamma \Rightarrow \varphi$

A Reset HA preproof is a proof if for every cycle $(\Theta_i ; \sigma_i \vdash \Gamma_i \Rightarrow \varphi_i)_{i \leq n}$ there exists a label $a \in \Theta_i$ for all $i \leq n$ and an a-Reset takes place along the cycle

$$
\cfrac{\cfrac{\boxed{a ; x \mapsto a \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)}}{ab ; x \mapsto ab \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)}\ \text{Reset}_a}{\cfrac{\Pi}{\cfrac{a ; x \mapsto a \vdash \varphi(0), (\forall x.\ (x \to (Sx)) \Rightarrow \varphi(x)}{\cfrac{\Pi}{\cfrac{\varepsilon ; x \mapsto \varepsilon \vdash \varphi(0), (\forall x. \varphi(x) \to \varphi(Sx)) \Rightarrow \varphi(x)}{\cfrac{\vdots}{\varepsilon ; x \mapsto \varepsilon \vdash \Rightarrow \varphi(0) \to (\forall x. \varphi(x) \to \varphi(Sx)) \to \forall x. \varphi(x)}}}}}}
$$

3. Nice Induction Orders

The Plan for Today



Nice Induction Orders

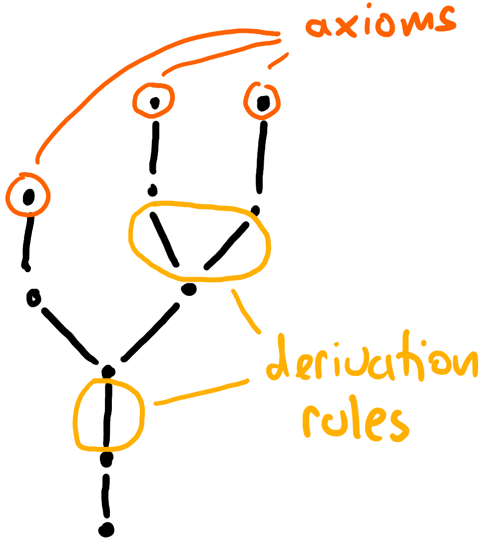
Warning: This is stricter than "plain" induction orders

Sequents: $\overbrace{\Lambda}^{\text{Sequence of Jump Targets}} \vdash \overbrace{\Gamma \Rightarrow \varphi}^{\text{HA sequent}}$

Jump Targets: $\hat{x} \longmapsto \underbrace{\Gamma' \Rightarrow \varphi'}_{\text{HA sequent}}$

$\hat{x} \in \bigcup \{ \{ x, \bar{x} \} \mid x \in FV(\Gamma, \varphi) \}$

Proofs:



Nice Induction Orders: Bud & Companion

$$\text{Comp}\quad \frac{\Delta, (x \mapsto \Gamma \Rightarrow \varphi) \vdash \Gamma \Rightarrow \varphi}{\Delta \vdash \Gamma \Rightarrow \varphi}$$

Jumps are initiated with x

$$\text{Bud}\quad \frac{}{\Delta, (\overline{x} \mapsto \Gamma \Rightarrow \varphi) \vdash \Gamma \Rightarrow \varphi}$$

Branches may only be closed with $\overline{x}$ Jumps

$\Rightarrow$ x inactive ; $\overline{x}$ active

Nice Induction Orders: Case

Turn x into $\bar{x}$ in all Jumps

$$\text{Case}\;\frac{\Delta \restriction x \vdash \Gamma(0) \Rightarrow \varphi(0) \qquad \overbrace{\Delta^{+x}} \vdash \Gamma(Sx) \Rightarrow \varphi(Sx)}{\Delta \vdash \Gamma(x) \Rightarrow \varphi(x)}$$

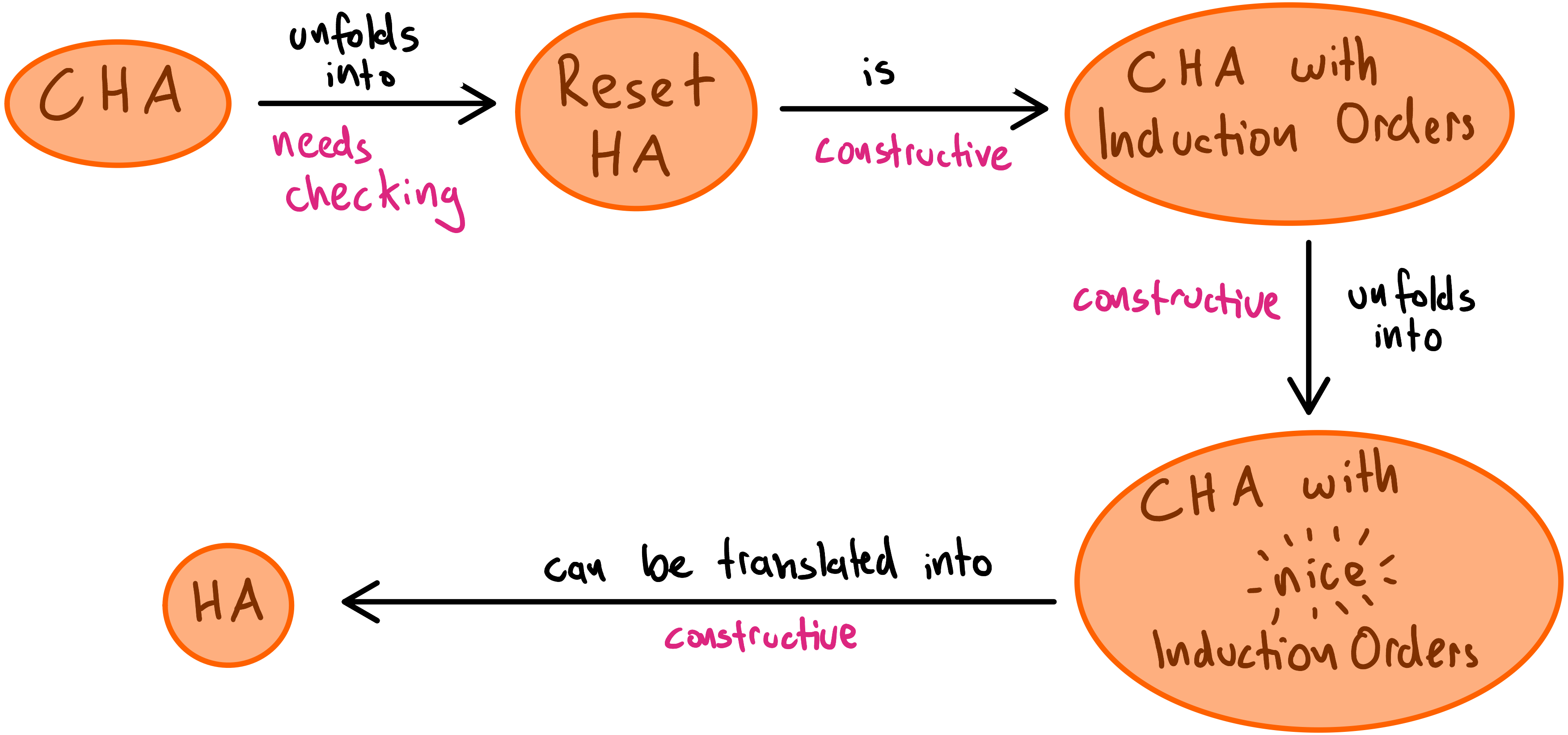
$$\left.\begin{aligned} (\hat{x} \mapsto \Gamma \Rightarrow \varphi), \Delta \restriction x &= \varepsilon \\ (q \mapsto \Gamma \Rightarrow \varphi), \Delta \restriction x &= (q \mapsto \Gamma \Rightarrow \varphi), (\Delta \restriction x) \\ \varepsilon \restriction x &= \varepsilon \end{aligned}\right\} \text{Prefix of } \Delta \text{ until the first instance of } \hat{x}$$

Nice Induction Orders: Example

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varepsilon \vdash \Rightarrow 0+0=0$}
\AxiomC{}
\LeftLabel{Bud}
\UnaryInfC{$\bar{x} \mapsto \Rrightarrow 0+x=x \vdash \Rrightarrow 0+x=x$}
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{}
\UnaryInfC{$\bar{x} \mapsto \Rrightarrow 0+x=x \vdash \Rrightarrow S(0+x)=Sx$}
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{}
\UnaryInfC{$\bar{x} \mapsto \Rrightarrow 0+x=x \vdash \Rrightarrow 0+Sx=Sx$}
\LeftLabel{Case}
\BinaryInfC{$x \mapsto \Rrightarrow 0+x=x \vdash \Rrightarrow 0+x=x$}
\LeftLabel{Comp}
\UnaryInfC{$\varepsilon \vdash \Rrightarrow 0+x=x$}
\end{prooftree}
$$

4. The Proof Translation

The Plan for Today



Translating NHA into HA

Claim: $NHA \vdash (\Lambda \vdash \Gamma \Rightarrow \varphi) \rightsquigarrow HA \vdash \Gamma, \underbrace{H_\Lambda}_{H_\Lambda = H\,\emptyset\,\Lambda\,\emptyset} \Rightarrow \varphi$

$$H\,\overbrace{V}^{\text{lower jump vars}}\,(x \mapsto \Gamma \Rightarrow \varphi, \Lambda)\,\overbrace{X}^{\text{lower hypotheses}} := H\,(V \cup \{x\})\,\Lambda\,(X \cup \{\lambda^g\})$$

$$H\,V\,(\bar{x} \mapsto \Gamma \Rightarrow \varphi, \Lambda)\,X := H\,(V \cup \{x\})\,\Lambda\,(X \cup \{\lambda\})$$

$$H\,V\,\varepsilon\,X := X$$

decreased $\longrightarrow$ unguarded

unguarded $\lambda := \overbrace{\forall y_1' \le y_1, \dots, y_n' \le y_n}^{\text{lower jump variables may decrease}} .\ \overbrace{\forall \bar{v}}^{\text{close free variables}}.((\bigwedge \Gamma \cup \overbrace{X}^{\text{hypotheses for lower jumps}} \to \varphi)[y_1/y_1', \dots, y_n/y_n'])$

guarded $\lambda^g := \underbrace{\forall x' < x}_{\text{guarded by } x}.(\lambda[x/x'])$

where $\{y_1, \dots, y_n\} = V \setminus \{x\}$ and $\bar{v} = FV(\Gamma, \varphi) \setminus (V \cup \{x\})$

Translating NHA → HA: Bud

$$\dfrac{}{\Lambda, (\bar{x} \mapsto \Gamma \Rightarrow \varphi) \vdash \Gamma \Rightarrow \varphi} \qquad \lambda = \forall \bar{y}' \leq \bar{q}. \forall \bar{v}. (\bigwedge \Gamma \cup H_\Lambda \to \varphi)[\tilde{q}/\bar{q}']$$

$$\forall L\,\dfrac{\to L\,\dfrac{\wedge R, Ax\,\dfrac{}{\Gamma, H_\Lambda \Rightarrow \bigwedge \Gamma \cup H_\Lambda} \qquad \dfrac{}{\Gamma, H_\Lambda, \varphi \Rightarrow \varphi}\,Ax}{\Gamma, H_\Lambda, (\bigwedge \Gamma \cup H_\Lambda \to \varphi) \Rightarrow \varphi}}{\Gamma, \underbrace{H_\Lambda, \lambda}_{H_{\Lambda, \bar{x} \mapsto \Gamma \Rightarrow \varphi}} \Rightarrow \varphi}$$

Translating NHA → HA: Comp

$$\dfrac{\Lambda, (x \mapsto \Gamma \Rightarrow \varphi) \vdash \Gamma \Rightarrow \varphi}{\Lambda \vdash \Gamma \Rightarrow \varphi}$$

$$\lambda = \forall \bar{y}' \leq \bar{y}. \forall \bar{v}. (\textstyle\bigwedge \Gamma \cup H_\Lambda \to \varphi)[\bar{y}/\bar{y}']$$

$$\lambda^g = \forall x' < x.\, \lambda[x/x']$$

$$\text{SInd}\ \dfrac{\forall x' < x. \varphi(x') \Rightarrow \varphi(x)}{\Rightarrow \forall x. \varphi(x)}$$

$$\text{Cut}\ \dfrac{\text{∀L}\ \dfrac{\dfrac{\text{As for Bud}}{\Gamma, H_\Lambda, \lambda \Rightarrow \varphi}}{\Gamma, H_\Lambda, \forall x. \lambda \Rightarrow \varphi} \qquad \text{wk}\ \dfrac{\dfrac{\text{∀R}\ \dfrac{\text{→L}\ \dfrac{\dfrac{\text{Inductive Hypotheses}}{\Gamma, H_\Lambda, \lambda^g \Rightarrow \varphi}}{\lambda^g \Rightarrow \bigwedge \Gamma \cup H_\Lambda \to \varphi}}{\forall x' < x. \lambda[x/x'] \Rightarrow \lambda}}{\Rightarrow \forall x. \lambda}}{\Gamma, H_\Lambda \Rightarrow \forall x. \lambda}}{\Gamma, H_\Lambda \Rightarrow \varphi}$$

Translating NHA → HA: Case

$$\frac{\Lambda\restriction x \vdash \Gamma(0) \Rightarrow \varphi(0) \qquad \Lambda^{+x} \vdash \Gamma(Sx) \Rightarrow \varphi(Sx)}{\Lambda \vdash \Gamma(x) \Rightarrow \varphi(x)}$$

① Invariants for $\varphi \neq x$

$$(\forall y < y.)\ \forall x' \leq Sx.\ \forall \bar{z}' \leq \bar{z}.\ \forall \bar{v}.\ \cdots \qquad \lambda^{(g)}(Sx)$$

$$(\forall y < y.)\ \forall x' \leq x.\ \forall \bar{z}' \leq \bar{z}.\ \forall \bar{v}.\ \cdots \qquad \lambda^{(g)}(x)$$

② Invariants for x

$$\forall x' < Sx.\ \forall \bar{q}' \leq \bar{q}.\ \forall \bar{v}.\ \cdots \qquad \lambda^{g}(Sx)$$

$$\forall \bar{q}' \leq \bar{q}.\ \forall \bar{v}.\ \cdots \qquad \lambda$$

Fact: $H_{\Lambda}(0) \subseteq H_{\Lambda\restriction x}(0) = H_{\Lambda\restriction x}$

No free instances of x!

$$\dfrac{\dfrac{\dfrac{\text{Inductive Hypothesis}}{\Gamma(0), H_{\Lambda\restriction x} \Rightarrow \varphi(0)}}{\Gamma(0), H_{\Lambda}(0) \Rightarrow \varphi(0)}\,\text{Wk} \qquad \dfrac{\dfrac{\text{Inductive Hypothesis}}{\Gamma(Sx), H_{\Lambda^{+x}} \Rightarrow \varphi(Sx)}}{\dfrac{\vdots}{\Gamma(Sx), H_{\Lambda}(Sx) \Rightarrow \varphi(Sx)}}}{\Gamma(x), H_{\Lambda}(x) \Rightarrow \varphi(x)}\,\text{Case}$$

Translating NHA → HA: Big Picture

Bud

$$\dfrac{}{\Lambda_3, \bar{x} \mapsto \Gamma \Rightarrow \varphi \vdash \Gamma \Rightarrow \varphi}$$

$\longrightarrow$ Close the proof using λ

$$\vdots$$

Case

$$\dfrac{\Lambda_2^{+x} \vdash \Gamma_2(Sx) \Rightarrow \varphi_2(Sx)}{\Lambda_2 \vdash \Gamma_2(x) \Rightarrow \varphi_2(x)}$$

$\longrightarrow$ Remove guard from λ^g to obtain $\lambda(x)$

$$\vdots$$

Comp

$$\dfrac{\Lambda, x \mapsto \Gamma \Rightarrow \varphi \vdash \Gamma \Rightarrow \varphi}{\Lambda \vdash \Gamma \Rightarrow \varphi}$$

$\longrightarrow$ Add guarded $\lambda^g = \forall x' < x.\, \lambda(x')$ to the hypotheses H_Λ

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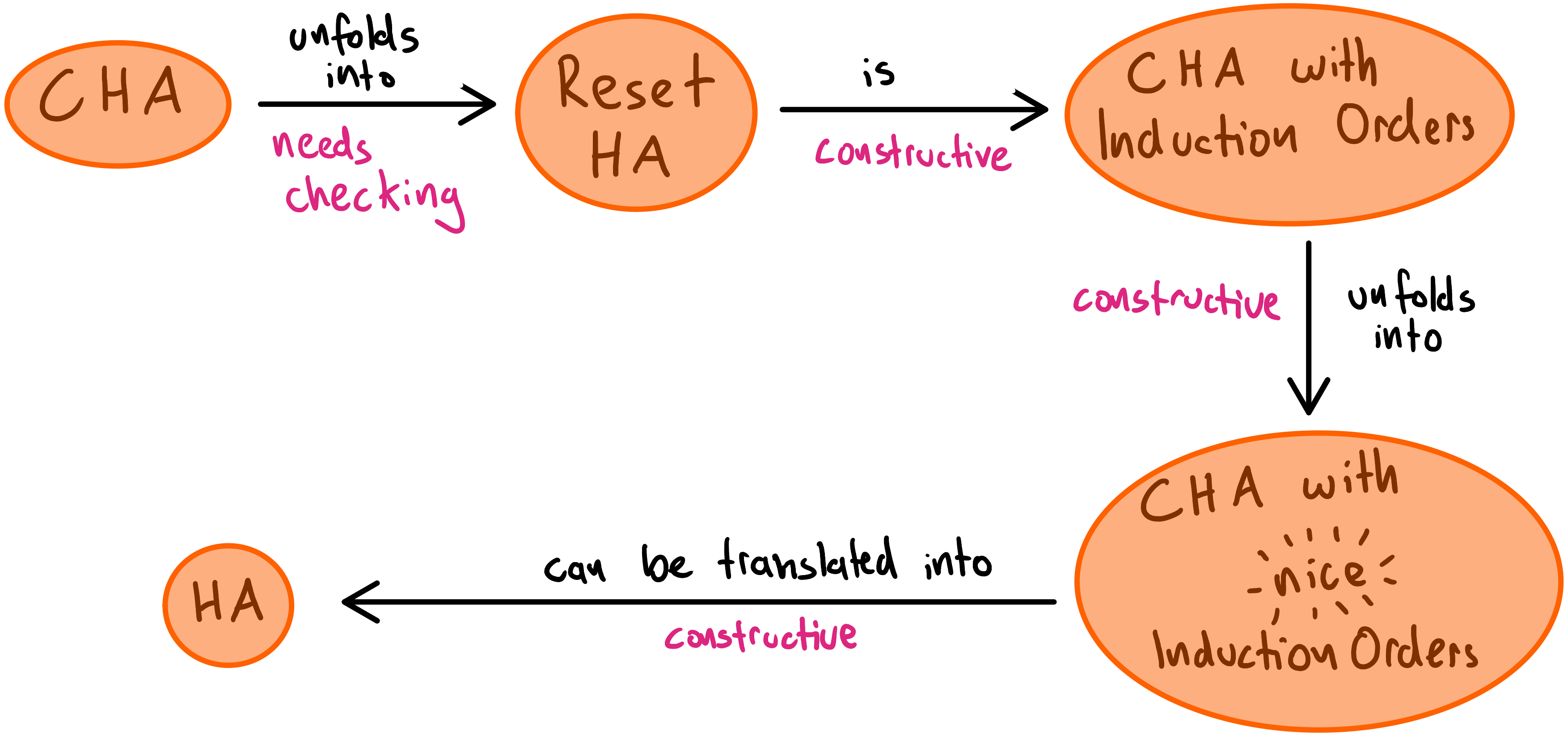

Related Work

Cyclic:

- On the Structure of Inductive Reasoning: Circular and Tree-Shaped Proofs in the μ Calculus — Sprenger, Dam 2003
- Finitary Proof Systems for Kozen's μ — Afshari, Leigh 2016

Arithmetical:

- Cyclic Arithmetic is equivalent to Peano Arithmetic — Simpson 2017
- On the logical complexity of cyclic arithmetic — Das 2020
- Equivalence of Intuitionistic Inductive Definitions and Intuitionistic Cyclic Proofs under Arithmetic — Berardi, Tatsuta 2017