A graphical calculus for linear categories

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Plan of the talk

Linear categories

Linear categories

Linear-nonlinear adjunctions

$$\begin{tikzcd} \mathcal{C} = (\mathcal{C}, \times, 1, \Rightarrow) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{L} = (\mathcal{L}, \otimes, \top, \multimap) \arrow[l, bend left] \end{tikzcd}$$

between a CCC $\mathcal{C}$ and an SMCC $\mathcal{L}$ are *ubiquitous*.

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Of our particular interest are **linear categories** – semantics of **ILL**

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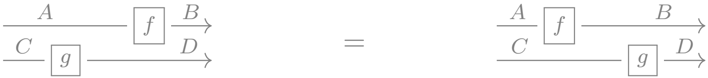

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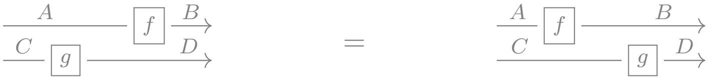

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We solve it by a graphical calculus that forms an *initial* linear category.

Plan of the talk

Example (1/2)

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$$\frac{\bot_{[x]}, \quad \bot_{[y]} \;\dashv\vdash\; \bot_{[\hat{y}]} \otimes \bot_{[\hat{x}]} \qquad\qquad \bot_{[z]} \otimes \bot_{[w]} \;\dashv\vdash\; \bot_{[\hat{w}]} \otimes \bot_{[\hat{z}]}}{}$$

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  where moves of F may be *initial*, *O* or *P*, and *joker* if on $\top$ or !;

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  in $\bar{t}_F$ that satisfy: o_1 is initial in F, $S_i \subseteq \bigcup_{j \leqslant i} \&_F(o_j)$, any segment on (e, V) is *compatible* with $\mathfrak{C}(t)_{(e,V)}$, and m is joker if it is O;

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- **Terms** – preterms with *no redundancy* for valid paths;

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  2. When $o \xrightarrow{S} p \xleftarrow{S'} o'$ with $o \neq o'$ in t, the P-move p is joker.

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- **$\approx$-eq.** – let $t \sim t' :: F$ if the !-expansion of a $\times$-slice of a logical term $t :: F$ is given from that of another $t' :: F$ by the replacement of the source of an edge whose target is joker,

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In this way, we replace derivable terms and their equality in type theory (given formally and inductively) with *geometric* objects and conditions.

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- **$\approx$-eq.** – let $t \sim t' :: F$ if the !-expansion of a $\times$-slice of a logical term $t :: F$ is given from that of another $t' :: F$ by the replacement of the source of an edge whose target is joker, and let $\approx$ be the least equivalence relation between logical terms containing $\sim$;
- **ℓ-terms** – an **ℓ-term** is a logical term *saturated* under $\approx$.

In this way, we replace derivable terms and their equality in type theory (given formally and inductively) with *geometric* objects and conditions.

Proposition (existence and uniqueness of ℓ-form)

Every logical preterm $t :: F$ has a unique ℓ-term $\ell t :: F$, or its **ℓ-form**.

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They must be distinguished, but without $\times$-states they coincide as

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Each ℓ-term $\ell t_0 :: F_0$ has a finite sequence $(\ell t_{i-1} :: F_{i-1} \to_\ell \ell t_i :: F_i)_{i=1}^n$ of ℓ-reduction, and any of these sequences satisfies

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where $\mathrm{nf}_\ell(\ell t_0 :: F_0) := \ell t_n :: F_n$ *is called the* ***normal form*** *of* $\ell t_0 :: F_0$*.*

By this theorem, the **ℓ-equivalence**

$$\ell t :: F \simeq_\ell \ell t' :: F' :\Leftrightarrow \mathrm{nf}_\ell(\ell t :: F) = \mathrm{nf}_\ell(\ell t' :: F')$$

between ℓ-terms is a well-defined equivalence.

Plan of the talk

The initiality theorem

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Theorem (a graphical initial linear category)

The ℓ-calculus forms an ***initial*** *linear category* $\mathrm{Cl}(\ell)$.

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- An object is a formula in intuitionistic linear logic;
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- The composition $A \xrightarrow{[\ell t::F]_\ell} B \xrightarrow{[\ell u::G]_\ell} C$, where

$$t :: F = t_A : T_A \dashv t_\Sigma : \Sigma \vdash t_B : T_B,$$
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is the ℓ-eq. class of the ℓ-form of the logical preterm

$$t_A : T_A \dashv t_\Sigma : \Sigma, t_B \cup u_B : \varsigma(T_B, T_B), u_\Pi : \Pi \vdash u_C : T_C;$$

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- The identity $\mathrm{id}_A : A \to A$ links pairs of corresponding moves.

Plan of the talk

The triple unit problem (1/2)

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Corollary (triple unit)

The initial linear category has just one morphism

$$((\top \multimap \top) \multimap \top) \multimap \top \to ((\top \multimap \top) \multimap \top) \multimap \top,$$

and just two morphisms

$$((X \multimap \top) \multimap \top) \multimap \top \rightrightarrows ((X \multimap \top) \multimap \top) \multimap \top.$$

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Proof.

For the first part, there is just one ℓ-term

$$\begin{array}{l} ((\top_{[v]} \multimap \top_{[t]}) \multimap \top_{[s]}) \multimap \top_{[r]} \dashv \\ \vdash ((\top_{[u]} \multimap \top_{[q]}) \multimap \top_{[p]}) \multimap \top_{[o]} \end{array}$$

The triple unit problem (2/2)

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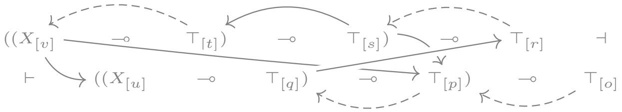
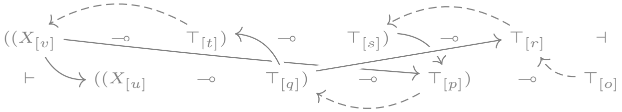
Proof (continued).

For the second part, there are just two ℓ-terms





□

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Proof.

Take the ℓ-terms that represent the canonical natural transformations whose parameters are given by propositional variables. Because they simply connect two occurrences of the same propositional variables, the coherence holds for these ℓ-terms.

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In any linear category, two parallel canonical natural transformations (where all the objects are parameters) are equal.

Proof.

Take the ℓ-terms that represent the canonical natural transformations whose parameters are given by propositional variables. Because they simply connect two occurrences of the same propositional variables, the coherence holds for these ℓ-terms. In essentially the same fashion, the coherence holds for the canonical natural transformations in $\mathrm{Cl}(\ell)$, and from this coherence the corollary follows by the initiality of $\mathrm{Cl}(\ell)$.