The axiomatization problem of dependence logic

Fan Yang

Utrecht University

LLAMA seminar
ILLC, Amsterdam, Feb. 22, 2023

$$\forall u\, \exists v\, \forall x\, \widehat{\exists y}\, \phi$$

Henkin Quantifiers (1961):

$$\begin{pmatrix} \forall u & \exists v \\ \forall x & \exists y \end{pmatrix} \phi$$

meaning $\exists f \exists g \forall u \forall x \phi(u, x, f(u)/v, g(x)/y)$

Independence-Friendly Logic (Hintikka and Sandu, 1989):

$$\forall u \exists v \forall x (\exists y/u) \phi$$

(Enderton, Walkoe, 1970)
first-order logic+Henkin quantifiers $\equiv$ existential second-order logic (ESO)
$\equiv$ independence-friendly logic

Dependence atoms (Väänänen 2007)

"x completely determines y"

$$=\!(x, y)$$

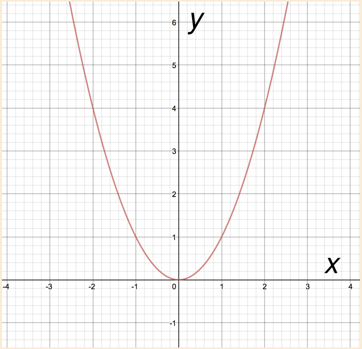
Dependence atoms (Väänänen 2007)

"*x* completely determines *y*"

$$=\!(x, y)$$

$\exists f$

Dependence atoms (Väänänen 2007)

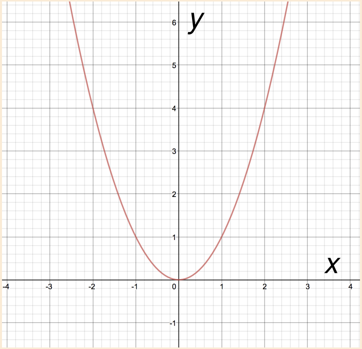


$y = f(x) = x^2$

"*x* completely determines *y*"

$$=(x, y)$$

$\exists f$

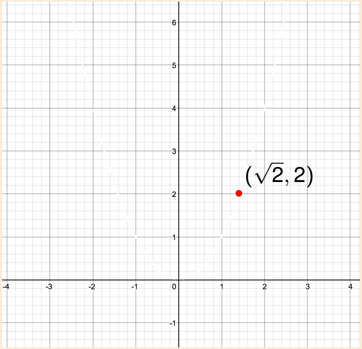


$y = f(x) = x^2$

	x	y	z
s	$\sqrt{2}$	2	0

Given a model M, and an assignment $s : \text{Var} \to M$,

$M \models_s$ "x completely determines y"??



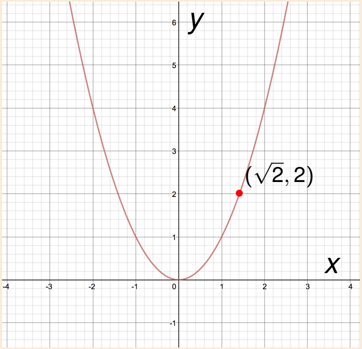
$y = f(x) = x^2$

	x	y	z
s	$\sqrt{2}$	2	0

Given a model M, and an assignment $s : \mathrm{Var} \to M$,

$M \models_s$ "x completely determines y"??

Dependence atoms (Väänänen 2007) & team semantics (Hodges 1997)



$y = f(x) = x^2$

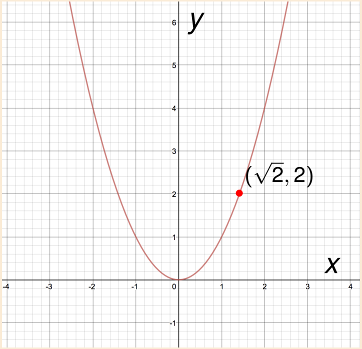
	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$M \models_X$ "x completely determines y" iff for all $s, s' \in X$,

$$s(x) = s'(x) \Longrightarrow s(y) = s'(y).$$



$y = f(x) = x^2$

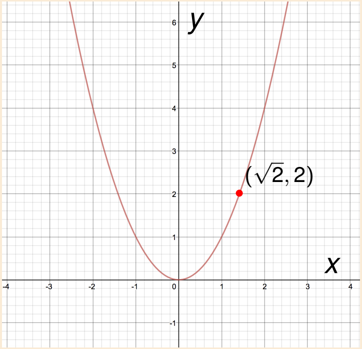
	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$M \models_X$ "x completely determines y" iff for all $s, s' \in X$,

$$s(x) = s'(x) \Longrightarrow s(y) = s'(y).$$



$y = f(x) = x^2$

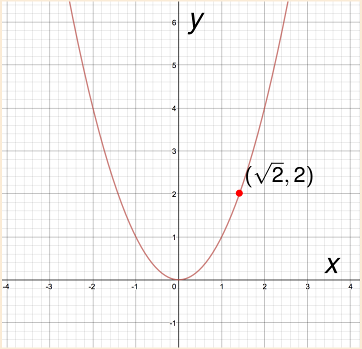
	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$M \models_X$ "x completely determines y" iff for all $s, s' \in X$,

$$s(x) = s'(x) \Longrightarrow s(y) = s'(y).$$



$y = f(x) = x^2$

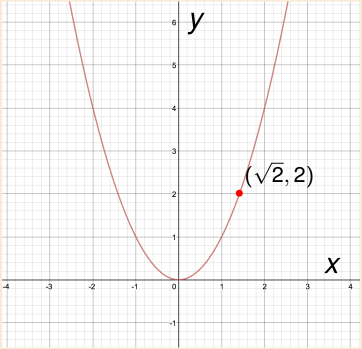
	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$M \models_X$ "x completely determines y" iff for all $s, s' \in X$,

$$s(x) = s'(x) \Longrightarrow s(y) = s'(y).$$



$y = f(x) = x^2$

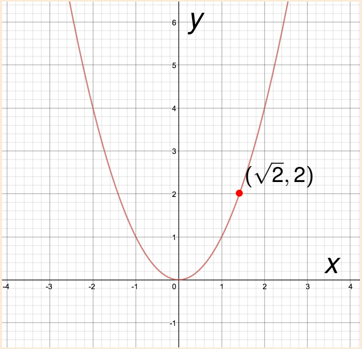
	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$M \models_X$ "x completely determines y" iff for all $s, s' \in X$,

$$s(x) = s'(x) \Longrightarrow s(y) = s'(y).$$



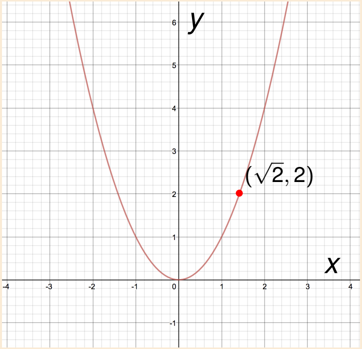
$y = f(x) = x^2$

	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$$M \models_X =\!(x, y) \text{ iff for all } s, s' \in X,$$
$$s(x) = s'(x) \implies s(y) = s'(y).$$



$y = f(x) = x^2$

	x	y	z
s_0	$\sqrt{2}$	2	0
s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
s_4	$-\sqrt{2}$	2	0

A team: a set of assignments $s : V \to M$

Given a model M, and a team X,

$$M \models_X =\!(\vec{x}, \vec{y}) \text{ iff for all } s, s' \in X,$$
$$s(\vec{x}) = s'(\vec{x}) \Longrightarrow s(\vec{y}) = s'(\vec{y}).$$

	x	y	z	v
s_0	c	d	a	a
s_1	c	d	a	b
s_2	a	e	c	c
s_3	a	e	c	d

- A team can be viewed as a relational database.
- Dependence atoms $=\!(\vec{x}, \vec{y})$ correspond exactly to functional dependencies $\vec{x} \rightarrow \vec{y}$ in database theory
- Armstrong's Axioms (1974) for functional dependencies:
  - $=\!(\vec{x}, \vec{x})$ (identity)
  - $=\!(\vec{x}\vec{y}, \vec{z})$ implies $=\!(\vec{y}\vec{x}, \vec{z})$ (commutativity)
  - $=\!(\vec{x}\vec{x}, \vec{y})$ implies $=\!(\vec{x}, \vec{y})$ (contraction)
  - $=\!(\vec{y}, \vec{z})$ implies $=\!(\vec{x}\vec{y}, \vec{z})$ (weakening)
  - $=\!(\vec{x}, \vec{y})$ and $=\!(\vec{y}, \vec{z})$ imply $=\!(\vec{x}, \vec{z})$ (transitivity)

- First-order logic (FO):

  $\alpha ::= t = t' \mid R\vec{t} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x\alpha \mid \forall x\alpha$

- Dependence logic (Väänänen 2007):

$$\text{first-order logic} + {=}(\vec{x}, \vec{y})$$

$\exists f$

- First-order logic (FO):

$$\alpha ::= t = t' \mid R\vec{t} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x\alpha \mid \forall x\alpha$$

- Dependence logic (Väänänen 2007):

$$\phi ::= \alpha \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\phi \mid \forall x\phi \mid {=}(\vec{x}, \vec{y})$$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

$x < y$

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X {=}(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

$x < y$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \Longrightarrow s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

$x < y$

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X {=}(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

$x < y$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X {=}(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

$x < y$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \Longrightarrow s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \;\&\; a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

$M \not\models_X x < y$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

$M \not\models_X x < y$

$M \not\models_X \neg x < y$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X {=}(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

$\alpha \vee \beta$

Team semantics

Let X be a team, i.e., a set of assignments $s : \text{Var} \to M$.

- $M \models_X =(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z		
3	4	5	α	
2	0	0	α	$\alpha \vee \beta$
1	2	3	α, β	
0	1	0	β	

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \Longrightarrow s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \;\&\; a \in M\}$.

x	y	z		
3	4	5	α	
2	0	0	α	$\alpha \vee \beta$
1	2	3	α, β	
0	1	0	β	

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v \alpha$ iff $\forall s \in X$: $M \models_s \exists v \alpha$
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v \alpha$ iff $\forall s \in X$: $M \models_s \exists v \alpha$
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

v: 0, 1, 2, 3, ⋮

M

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =\!(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \Longrightarrow s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v\alpha$ iff $\forall s \in X$: $M \models_s \exists v\alpha$
- $M \models_X \forall v\phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

v
0
1
2
3
⋮

M

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X {=}(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v \phi$ iff $M \models_{X(F/v)} \phi$ for some $F : X \to \wp(M) \setminus \{\emptyset\}$, where
  $$X(F/v) = \{s(a/v) \mid s \in X \text{ and } a \in F(s)\}.$$
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \;\&\; a \in M\}$.

x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

v
0
1
2
3
⋮

M

Team semantics

Let X be a team, i.e., a set of assignments $s : \text{Var} \to M$.

- $M \models_X =(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v \phi$ iff $M \models_{X(F/v)} \phi$ for some $F : X \to \wp(M) \setminus \{\emptyset\}$, where
$$X(F/v) = \{s(a/v) \mid s \in X \text{ and } a \in F(s)\}.$$
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z	v
3	4	5	0
2	0	0	1
1	2	3	2
0	1	0	3
			⋮

M

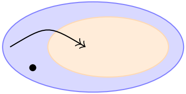
Team semantics

Let X be a team, i.e., a set of assignments $s : \text{Var} \to M$.

- $M \models_X =(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \Longrightarrow s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v \phi$ iff $M \models_{X(F/v)} \phi$ for some $F : X \to \wp(M) \setminus \{\emptyset\}$, where
$$X(F/v) = \{s(a/v) \mid s \in X \text{ and } a \in F(s)\}.$$
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \;\&\; a \in M\}$.

Empty team property: $M \models_\emptyset \phi$.

Downward closure: $M \models_X \phi$ and $Y \subseteq X \Longrightarrow M \models_Y \phi$.

$|M| = \infty$ iff $\exists f$:  iff $M \models \phi_\infty$

- An existential second-order (ESO) sentence:

  $\exists f \exists v \forall x_0 \forall x_1 \big((f(x_0) = f(x_1) \to x_0 = x_1) \wedge (f(x_0) \neq v)\big)$

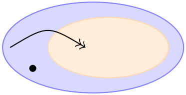
- An FO(=(...))-sentence:

$$\phi_\infty := \exists v \forall x \exists y (=(x,y) \wedge =(y,x) \wedge (v \neq y))$$

$\exists f$ $\quad$ f is 1-1

In general, ESO $\Longrightarrow$ FO(=(...)):

$$\exists f \forall \vec{x} \alpha(\vec{x}, f(\vec{x_i})) \equiv \forall \vec{x} \exists y (=(\vec{x_i}, y) \wedge \alpha(\vec{x}, y)).$$

$|M| = \infty$ iff $\exists f$:  iff $M \models \phi_\infty$

- An existential second-order (ESO) sentence:

  $\exists f \exists v \forall x_0 \forall x_1 \big((f(x_0) = f(x_1) \to x_0 = x_1) \wedge (f(x_0) \neq v) \big)$

- An FO(=(...))-sentence:

$$\phi_\infty := \exists v \forall x \exists y (=\!(x, y) \wedge =\!(y, x) \wedge (v \neq y))$$

$\exists f$ f is 1-1

In general, ESO $\Longrightarrow$ FO(=(...)):

$$\exists \vec{f} \forall \vec{x} \alpha(\vec{x}, f_1(\vec{x_1}), \ldots, f_k(\vec{x_k})) \equiv \forall \vec{x} \exists \vec{y} (\bigwedge_{i=1}^{k} =\!(\vec{x_i}, y_i) \wedge \alpha(\vec{x}, \vec{y})).$$

Theorem (Väänänen 2007)

For any ESO*-sentence* ϕ*, there is an* FO(=(...))*-sentence* χ_ϕ *such that*

$$M \models \phi \iff M \models \chi_\phi;$$

and vice versa.

Proof (Idea).

- ESO $\Longrightarrow$ FO(=(...)):

$$\exists \vec{f} \forall \vec{x} \alpha(\vec{x}, f_1(\vec{x_1}), \ldots, f_k(\vec{x_k})) \equiv \forall \vec{x} \exists \vec{y} (\bigwedge_{i=1}^{k} =(\vec{x_i}, y_i) \wedge \alpha(\vec{x}, \vec{y})).$$

- FO(=(...)) $\Longrightarrow$ ESO(R) : [Rmk: free variables $\vec{v} \rightsquigarrow$ teams $\rightsquigarrow$ relations]

  E.g., $\chi_{\phi \vee \psi}(R) = \exists S \exists T (\forall \vec{v}(R\vec{v} \leftrightarrow S\vec{v} \vee T\vec{v}) \wedge \chi_\phi(S) \wedge \chi_\psi(T)).$ □

Recall: $M \models_X \phi \vee \psi$ iff $\exists Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.

Theorem (Väänänen 2007)

For any ESO*-sentence* ϕ*, there is an* FO(=(...))*-sentence* χ_ϕ *such that*

$$M \models \phi \iff M \models \chi_\phi;$$

and vice versa.

Proof (Idea).

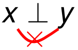
- ESO $\Longrightarrow$ FO(=(...)): Normal Form

$$\exists \vec{f} \forall \vec{x} \alpha(\vec{x}, f_1(\vec{x_1}), \ldots, f_k(\vec{x_k})) \equiv \boxed{\forall \vec{x} \exists \vec{y} \left(\bigwedge_{i=1}^{k} =(\vec{x_i}, y_i) \wedge \alpha(\vec{x}, \vec{y}) \right)}.$$

- FO(=(...)) $\Longrightarrow$ ESO(R) : [Rmk: free variables $\vec{v} \rightsquigarrow$ teams $\rightsquigarrow$ relations]
  E.g., $\chi_{\phi\vee\psi}(R) = \exists S \exists T \big(\forall \vec{v}(R\vec{v} \leftrightarrow S\vec{v} \vee T\vec{v}) \wedge \chi_\phi(S) \wedge \chi_\psi(T)\big)$. □

Recall: $M \models_X \phi \vee \psi$ iff $\exists Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.

- $M \models_X \vec{x} \perp \vec{y}$ iff for all $s, s' \in X$, there exists $s'' \in X$ such that $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$.



	x	y	z
s	a	b	e
s'	c	d	d
	c	b	e
s''	a	d	a

- $\text{ESO}(R^{\downarrow}) \equiv \text{FO}(=(\ldots)) \lneq \text{FO}(\perp) \equiv \text{ESO}(R)$
  (Kontinen, Väänänen 2009) (Galliani 2012)
- Over sentences, $\text{FO}(\perp) \equiv \text{FO}(=(\ldots)) \equiv \text{ESO}$ (Grädel, Väänänen 2013)

Cor. Neither FO(=(...)) nor FO(⊥) is (effectively) axiomatizable.

Partial axiomatizations

$$\Gamma \vdash \theta \iff \Gamma \models \theta$$

Two approaches:

- Consider weaker fragments of the logics: (Kontinen, Y. 2022) C.f. (Baltag, van Benthem 2021)
- Consider weaker consequence relations $\vdash \subseteq \wp(\mathit{Form}) \times \mathit{Form}_0$, where $\mathit{Form}_0 \subseteq \mathit{Form}$

Theorem (Kontinen, Väänänen 2013 & Hannula 2015)

There are (sound) natural deduction systems for FO(=(...)) *and* FO($\perp$) *such that*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha$$

for any set Γ *of sentences and any first-order sentence* α.

Axiomatizing essentially first-order consequences

Theorem (Kontinen, Väänänen 2013 & Hannula 2015 & Y. 2019)

There are (sound) natural deduction systems for FO(=(...)) *and* FO($\perp$) *such that*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha$$

for any set Γ *of formulas and any first-order formula* α.

Axiomatizing essentially first-order consequences

Theorem (Kontinen, Väänänen 2013 & Hannula 2015 & Y. 2019)

There are (sound) natural deduction systems for FO(=(...)) *and* FO($\perp$) *such that*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha$$

for any set Γ *of formulas and any essentially first-order / negatable formula* α.

Axiomatizing essentially first-order consequences

Theorem (Kontinen, Väänänen 2013 & Hannula 2015)

There are (sound) natural deduction systems for FO(=(...)) *and* FO(⊥) *such that*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha$$

for any set Γ *of sentences and any first-order sentence* α.

Theorem (Kontinen, Väänänen 2013 & Hannula 2015)

There are (sound) natural deduction systems for FO(=(...)) *and* FO(⊥) *such that*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha$$

for any set Γ *of sentences and any first-order sentence* α*.*

Observation: Assuming $\Gamma \nvdash \alpha$, we want to show $\Gamma \not\models \alpha$, which is equivalent to $\{\chi_\gamma \mid \gamma \in \Gamma\} \not\models \chi_\alpha \{\chi_\gamma \mid \gamma \in \Gamma\}, \neg\chi_\alpha \not\models \bot$, where χ_γ and χ_α are the ESO-translations of γ and α respectively, and χ_α is equivalent to a first-order formula. It then suffices to find a τ-model M for the set

$$\{\chi_\gamma \mid \gamma \in \Gamma\} \cup \{\neg\chi_\alpha\} = \{\exists \vec{R}_\gamma \gamma^* \mid \gamma \in \Gamma\} \cup \{\neg\chi_\alpha\}.$$

This can be reduced to finding a $\tau(\langle \vec{R}_\gamma \rangle_{\gamma\in\Gamma})$-model $(M, \langle \vec{R}_\gamma^M \rangle_{\gamma\in\Gamma})$ for the set

$$\{\gamma^* \mid \gamma \in \Gamma\} \cup \{\neg\chi_\alpha\}$$

of first-order sentences, which can in principle be done within first-order logic.

$$\frac{\exists y \forall x \phi(x, y, \vec{v})}{\forall x \exists y (=(\vec{v}, y) \land \phi)} \text{ DepI}$$

$$\frac{\phi}{\phi \lor \psi} \lor\text{I} \qquad \frac{\phi}{\psi \lor \phi} \lor\text{I} \qquad \frac{\phi \lor \psi \quad \begin{matrix}[\phi] \\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \alpha\end{matrix}}{\alpha} \lor\text{E}$$

α is first-order

Example: $=(x, y) \lor =(x, y) \not\models =(x, y)$.

x	y
a	b
a	c

Prop (conservative extension). For any set $\Delta \cup \{\alpha\}$ of FO-formulas,

$$\Delta \vdash_{\text{FO}} \alpha \iff \Delta \vdash_{\text{FO}(=(\dots))} \alpha.$$

Partial completeness theorem

Theorem (Kontinen, Väänänen 2013)

For any set Γ *of* $\mathrm{FO}(=(\ldots))$*-sentences and any first-order sentence* α*,*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha.$$

Proof sketch (Kontinen, Väänänen 2013)

1. Every FO(=(...))-sentence is (semantically and provably) equivalent to a formula of the form $\phi = \forall\vec{x}\exists\vec{y}(\bigwedge_{i\in I} {=}(\vec{x}_i, y_i) \wedge \alpha)$.
2. There is a first-order infinitary sentence γ_ϕ (called the *game expression* of ϕ) s.t. for any countable model M,
$$M \models \phi \iff M \models \gamma_\phi.$$
3. γ_ϕ can be approximated by some first-order sentences γ_ϕ^n ($n < \omega$) in the sense that for any recursively saturated (or finite) model M,
$$M \models \gamma_\phi \iff M \models \gamma_\phi^n \text{ for all } n < \omega.$$
We also have $\phi \vdash \gamma_\phi^n$.
4. For any set $\Gamma \cup \{\alpha\}$ of FO(=(...))-sentences with α first-order,
$\Gamma \nvdash \alpha \implies \Gamma^* \nvdash \alpha$, where $\Gamma^* = \{\gamma_\phi^n \mid \phi \in \Gamma \text{ and } n < \omega\}$
$\implies \Gamma^* \nvdash_{\mathrm{FO}} \alpha$
$\implies \Gamma^* \cup \{\neg\alpha\}$ has a finite or cnt. and rec. sat. model M
$\implies \Gamma \cup \{\neg\alpha\}$ has a model M, and thereby $\Gamma \not\models \alpha$. □

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=(x,y) \wedge \alpha(x,y,v))$.

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

X

	x	v	y
0			
1			
2			
3			
⋮			

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

$M \models_X \quad =(x, y)$

$\qquad \alpha$

	x	v	y
0			
1			
2			
3			
⋮			

Consider $\phi = \forall xv \exists y(=(x,y) \wedge \alpha(x,y,v))$.

$M \models_X \quad =\!(x, y)$

α

	x	v	y
0			
1			
2			
3			
⋮			

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

$M \models_X$ $=(x, y)$
α

	x	v	y
0			
1			
2			
3			
⋮			

Consider $\phi = \forall x v \exists y(=(x,y) \wedge \alpha(x,y,v))$.

$M \models_X \underbrace{=(x,y)}_{\alpha}$

	x	v	y
0			
1			
2			
3			
⋮			

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

$M \models_X$ $=(x, y)$
α

	x	v	y
0			
1			
2			
3			
⋮			

Consider $\phi = \forall xv \exists y(=\!(x, y) \wedge \alpha(x, y, v))$.

$M \models_X \quad =\!(x, y)$

α

	x v	y
0		
1		
2		
3		
⋮		

$$\begin{aligned}\gamma_\phi = \forall x_0 v_0 \exists y_0 \big(\alpha(x_0 v_0 y_0) \wedge \\ \forall x_1 v_1 \exists y_1 \big(\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \wedge \\ \forall x_2 v_2 \exists y_2 \big(\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \\ \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \wedge \\ \ldots \quad \ldots \big)\big)\big)\end{aligned}$$

Consider $\phi = \forall xv \exists y(=(x,y) \wedge \alpha(x,y,v))$.

$M \models_X \; =(x,y)$
α

	x	v	y
0			
1			
2			
3			
⋮			

$$\begin{aligned}\gamma_\phi = {} & \forall x_0 v_0 \exists y_0 \big(\alpha(x_0 v_0 y_0) \wedge \\ & \forall x_1 v_1 \exists y_1 \big(\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \to y_0 = y_1) \wedge \\ & \forall x_2 v_2 \exists y_2 \big(\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \to y_0 = y_2) \\ & \qquad \wedge (x_1 = x_2 \to y_1 = y_2) \wedge \\ & \qquad \qquad \cdots \quad \ldots \big)\big)\big)\end{aligned}$$

Consider $\phi = \forall xv \exists y(=(x,y) \wedge \alpha(x,y,v))$.

$$M \models_X \underset{\alpha}{=}(x,y)$$

	x	v	y
0			
1			
2			
3			
$\vdots$			

$$\gamma_\phi = \forall x_0 v_0 \exists y_0 (\alpha(x_0 v_0 y_0) \wedge \forall x_1 v_1 \exists y_1 (\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \to y_0 = y_1) \wedge \forall x_2 v_2 \exists y_2 (\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \to y_0 = y_2) \wedge (x_1 = x_2 \to y_1 = y_2) \wedge \dots \dots)))$$

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

$M \models_X \; =(x, y)$

α

	x	v	y
0			
1			
2			
3			
⋮			

$$\begin{aligned}\gamma_\phi = & \forall x_0 v_0 \exists y_0 (\alpha(x_0 v_0 y_0) \wedge \\ & \forall x_1 v_1 \exists y_1 (\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \to y_0 = y_1) \wedge \\ & \forall x_2 v_2 \exists y_2 (\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \to y_0 = y_2) \\ & \qquad \wedge (x_1 = x_2 \to y_1 = y_2) \wedge \\ & \qquad \ldots \quad \ldots)))\end{aligned}$$

Consider $\phi = \forall xv\exists y(=(x, y) \wedge \alpha(x, y, v))$.

$M \models_X \ =(x, y)$
α

	x	v	y
0			
1			
2			
3			
⋮			

$$\gamma_\phi = \forall x_0 v_0 \exists y_0 (\alpha(x_0 v_0 y_0) \wedge \\ \forall x_1 v_1 \exists y_1 (\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \wedge \\ \forall x_2 v_2 \exists y_2 (\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \\ \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \wedge \\ \dots \quad \dots)))$$

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

$$M \models_X \; =(x, y) \quad \alpha$$

	x	v	y
0			
1			
2			
3			
⋮			

$$\begin{aligned}\gamma_\phi = \; & \forall x_0 v_0 \exists y_0 (\alpha(x_0 v_0 y_0) \wedge \\ & \forall x_1 v_1 \exists y_1 (\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \to y_0 = y_1) \wedge \\ & \forall x_2 v_2 \exists y_2 (\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \to y_0 = y_2) \\ & \qquad \wedge (x_1 = x_2 \to y_1 = y_2) \wedge \\ & \qquad\qquad \ldots \quad \ldots)))\end{aligned}$$

Consider $\phi = \forall xv\exists y(=(x,y) \land \alpha(x,y,v))$.

$$M \models_X \underset{\alpha}{\quad} =(x,y)$$

	x	v	y
0			
1			
2			
3			
⋮			

$$\begin{aligned}\gamma_\phi = \forall x_0 v_0 \exists y_0 (\alpha(x_0 v_0 y_0) \land \\ \forall x_1 v_1 \exists y_1 (\alpha(x_1 v_1 y_1) \land (x_0 = x_1 \to y_0 = y_1) \land \\ \forall x_2 v_2 \exists y_2 (\alpha(x_2 v_2 y_2) \land (x_0 = x_2 \to y_0 = y_2) \\ \land (x_1 = x_2 \to y_1 = y_2) \land \\ \dots \quad \dots)))\end{aligned}$$

Consider $\phi = \forall x v \exists y(=\!(x, y) \wedge \alpha(x, y, v))$.

$$M \models_X \underset{\alpha}{=\!(x, y)}$$

	x	v	y
0			
1			
2			
3			
⋮			

$$\begin{aligned}\gamma_\phi = {} & \forall x_0 v_0 \exists y_0 \big(\alpha(x_0 v_0 y_0) \wedge \\ & \forall x_1 v_1 \exists y_1 \big(\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \wedge \\ & \forall x_2 v_2 \exists y_2 \big(\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \\ & \qquad \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \wedge \\ & \qquad \ldots \quad \ldots \big)\big)\big)\end{aligned}$$

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=\!(x,y) \wedge \alpha(x,y,v))$.

$M \models_X \underset{\alpha}{=\!(x,y)}$

	x	v	y
0			
1			
2			
3			
$\vdots$			

$$\begin{aligned}\gamma_\phi = {} & \forall x_0 v_0 \exists y_0 (\alpha(x_0 v_0 y_0) \wedge \\ & \forall x_1 v_1 \exists y_1 (\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \to y_0 = y_1) \wedge \\ & \forall x_2 v_2 \exists y_2 (\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \to y_0 = y_2) \\ & \qquad \wedge (x_1 = x_2 \to y_1 = y_2) \wedge \\ & \qquad\qquad \ldots \quad \ldots)))\end{aligned}$$

Thm. If M is countable, then $M \models \phi \iff M \models \gamma_\phi$.

The game expression in FO(=(...))

Consider $\phi = \forall xv \exists y(=(x, y) \wedge \alpha(x, y, v))$.

$M \models_X \quad =(x, y)$

α

	x	v	y
0			
1			
2			
3			
⋮			

$$
\begin{aligned}
\gamma_\phi = \forall x_0 v_0 \exists y_0 \big(\alpha(x_0 v_0 y_0) \wedge \\
\forall x_1 v_1 \exists y_1 \big(\alpha(x_1 v_1 y_1) \wedge (x_0 = x_1 \to y_0 = y_1) \wedge \\
\forall x_2 v_2 \exists y_2 \big(\alpha(x_2 v_2 y_2) \wedge (x_0 = x_2 \to y_0 = y_2) \\
\wedge (x_1 = x_2 \to y_1 = y_2) \wedge \\
\ldots \quad \ldots \big)\big)\big)
\end{aligned}
$$

γ_ϕ^2

Thm. If M is countable, then $M \models \phi \iff M \models \gamma_\phi$.

Thm. If M is recursively saturated (or finite), then

$$M \models \gamma_\phi \iff M \models \gamma_\phi^n \text{ for all } n < \omega.$$

Theorem (Kontinen, Väänänen 2013)

For any set Γ *of* FO(=(...))*-sentences and any first-order sentence* α*,*

$$\Gamma \models \alpha \iff \Gamma \vdash \alpha.$$

Application examples:

- $\phi_\infty \vdash \phi_{\geq n}$ for all $n \in \omega$, where $\phi_{\geq n}$ states "the model has $\geq n$ elements"
- $\phi_{nwf} \vdash \exists x_1 \dots \exists x_n (x_1 < \dots < x_n)$, where ϕ_{nwf} states "$<$ is not well-founded"

Next: Generalization using "negation".

Closure under negation

FO($=(\ldots)$) $\equiv$ FO($\perp$) $\equiv$ ESO are not closed under (weak) classical negation, where

- Classical negation: $M \models_X \sim \phi$ iff $M \not\models_X \phi$
- Weak classical negation: $M \models_X \dot{\sim} \phi$ iff either $X = \emptyset$ or $M \not\models_X \phi$

Fact: Since $M \models_\emptyset \phi$, $M \not\models_\emptyset \sim \phi$. Thus, $\sim \phi \not\equiv \psi$ for any ψ in FO($=(\ldots)$) or FO($\perp$).

Def. We say that ϕ is negatable in L if $\dot{\sim} \phi \equiv \psi$ for some L-formula ψ.

Fact: In general, $\neg\alpha \not\equiv \dot{\sim} \alpha$ for arbitrary formula α in FO.
The negation $\neg\phi$ is not well-defined for arbitrary formulas ϕ in FO($=(\ldots)$) and FO($\perp$).

Theorem (Y. 2019)

Let ϕ be a formula of FO($\bot$). *TFAE:*

1. *ϕ is negatable in* FO($\bot$).
2. *The* ESO*-translation $\chi_\phi(R)$ of ϕ is equivalent to a first-order formula.*

Proof. By the Interpolation Theorem of first-order logic. □

Fact: The class of negatable formulas is undecidable.

Example:

- First-order formulas α are negatable in FO($\bot$), as $\chi_\alpha(R) = \forall\vec{x}(R(\vec{x}) \to \alpha(\vec{x}))$. In particular, $\dot{\sim}\alpha(\vec{v}) \equiv \exists\vec{x}(\vec{x} \subseteq \vec{v} \wedge \neg\alpha(\vec{x}))$, where $\vec{x} \subseteq \vec{v}$ is the inclusion atom
- Dependence and independence atoms are negatable in FO($\bot$), and their weak classical negations are uniformly definable.
- There is a hierarchy of negatable formulas θ in FO($\bot$) for which $\dot{\sim}\theta$ can be defined uniformly. (Y. 2019)

Theorem (Y. 2019)

Let $\Gamma \cup \{\theta\}$ be a set of formulas of L, where θ is negatable in L. We have that $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$$\begin{aligned}
\Gamma \models \theta &\implies \Gamma, \dot\sim\theta \models \bot, \text{ where } \dot\sim\theta \text{ stands for the } L\text{-formula } \phi \equiv \dot\sim\theta \\
&\implies \exists\vec{x}(\bigwedge\Gamma \wedge \dot\sim\theta) \models \bot, \text{ where } \vec{x} \text{ lists all free variables} \\
&\implies \exists\vec{x}(\bigwedge\Gamma \wedge \dot\sim\theta) \vdash_L \bot \quad \text{(by the completeness thm)} \\
&\implies \Gamma, \dot\sim\theta \vdash_L \bot \quad \text{(by the rule } \exists\text{I)} \\
&\implies \Gamma \vdash^*_L \theta. \quad \square
\end{aligned}$$

New rule:

$$\dfrac{\begin{matrix}[\dot\sim\theta] \\ \vdots \\ \bot\end{matrix}}{\theta}\,\text{RRA}$$

- Dependence and independence atoms are negatable in FO($\bot$).
  ⇝ Armstrong's axioms for functional dependencies, and Geiger-Paz-Pearl axioms for independence are derivable in FO($\bot$).
  ⇝ some facts concerning independence notions in quantum theory are derivable. (Abramsky, Puljujärvi, Väänänen 2021)

Theorem (Y. 2019)

Let $\Gamma \cup \{\theta\}$ be a set of formulas of L, where θ is negatable in L. We have that $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$$\begin{aligned}
\Gamma \models \theta &\implies \Gamma, \dot\sim\theta \models \bot, \text{ where } \dot\sim\theta \text{ stands for the } L\text{-formula } \phi \equiv \dot\sim\theta \\
&\implies \exists\vec{x}(\bigwedge\Gamma \wedge \dot\sim\theta) \models \bot, \text{ where } \vec{x} \text{ lists all free variables} \\
&\implies \exists\vec{x}(\bigwedge\Gamma \wedge \dot\sim\theta) \vdash_L \bot \quad \text{(by the completeness thm)} \\
&\implies \Gamma, \dot\sim\theta \vdash_L \bot \quad \text{(by the rule } \exists\text{I)} \\
&\implies \Gamma \vdash^*_L \theta. \quad \square
\end{aligned}$$

New rule:

$$\dfrac{\begin{array}{c}[\dot\sim\theta] \\ \vdots \\ \bot\end{array}}{\theta}\,\text{RRA}$$

- Arrow's Impossibility Theorem is derivable in the system of FO(⊥), i.e., $\Gamma \vdash \phi_{\text{dictator}}$ or $\Gamma, \dot\sim\phi_{\text{dictator}} \vdash \bot$. (Pacuit, Y. 2016)
- If $\Gamma \models \bot$, then $\Gamma \vdash \bot$. (Cf. Hintikka 1996)

Theorem

Let $\Gamma \cup \{\theta\}$ *be a set of formulas of* L, *where* θ *is negatable in* L. *We have that* $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$$
\begin{aligned}
\Gamma \models \theta &\implies \Gamma, \dot\sim\theta \models \bot, \text{ where } \dot\sim\theta \text{ stands for the } L\text{-formula } \phi \equiv \dot\sim\theta \\
&\implies \exists\vec{x}(\textstyle\bigwedge\Gamma \wedge \dot\sim\theta) \models \bot, \text{ where } \vec{x} \text{ lists all free variables} \\
&\implies \exists\vec{x}(\textstyle\bigwedge\Gamma \wedge \dot\sim\theta) \vdash_L \bot \quad \text{(by the completeness thm)} \\
&\implies \Gamma, \dot\sim\theta \vdash_L \bot \qquad\qquad \text{(by the rule } \exists\text{I)} \\
&\implies \Gamma \vdash^*_L \theta. \quad \square
\end{aligned}
$$

New rule:

$$
\cfrac{\begin{matrix}[\dot\sim\theta] \\ \vdots \\ \bot\end{matrix}}{\theta}\ \text{RRA}
$$

Theorem

Let $\Gamma \cup \{\theta\}$ be a set of formulas of L*, where θ is negatable in* L*. We have that* $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$\Gamma \models \theta \implies \Gamma, \dot\sim\theta \models \bot$, where $\dot\sim\theta$ stands for the L-formula $\phi \equiv \dot\sim\theta$

$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \dot\sim\theta) \models \bot$, where $\vec{x}$ lists all free variables

$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \dot\sim\theta) \vdash_L \bot$ (by the completeness thm)

$\implies \Gamma, \dot\sim\theta \vdash_L \bot$ (by the rule ∃I)

$\implies \Gamma \vdash^*_L \theta$. □

New rule:

$$\dfrac{\begin{matrix}[\dot\sim\theta] \\ \vdots \\ \bot\end{matrix}}{\theta}\ \text{RRA}$$

Even more general, with essentially negatable formulas

Theorem

Let $\Gamma \cup \{\theta\}$ be a set of formulas of L*, where θ is negatable in* L*. We have that* $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$\Gamma \models \theta \implies \Gamma, \theta^{\dot\sim} \models \bot$, where $\dot\sim\theta$ stands for the L-formula $\phi \equiv \dot\sim\theta$

$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \theta^{\dot\sim}) \models \bot$, where $\vec{x}$ lists all free variables

$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \theta^{\dot\sim}) \vdash_L \bot$ (by the completeness thm)

$\implies \Gamma, \theta^{\dot\sim} \vdash_L \bot$ (by the rule $\exists$I)

$\implies \Gamma \vdash^*_L \theta$. $\square$

New rule:

$$\dfrac{\begin{matrix}[\dot\sim\theta] \\ \vdots \\ \bot\end{matrix}}{\theta}\ \text{RRA}$$

Theorem

Let $\Gamma \cup \{\theta\}$ *be a set of formulas of* L, *where* θ *is essentially negatable in* L*. We have that* $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$$\Gamma \models \theta \implies \Gamma, \theta^{\dot\sim} \models \bot$$, where $\dot\sim\theta$ stands for the L-formula $\phi \equiv \dot\sim\theta$

$$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \theta^{\dot\sim}) \models \bot$$, where $\vec{x}$ lists all free variables

$$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \theta^{\dot\sim}) \vdash_L \bot \quad \text{(by the completeness thm)}$$

$$\implies \Gamma, \theta^{\dot\sim} \vdash_L \bot \quad \text{(by the rule } \exists\text{I)}$$

$$\implies \Gamma \vdash^*_L \theta. \qquad \square$$

New rule:

$$\dfrac{\begin{matrix}[\theta^{\dot\sim}] \\ \vdots \\ \bot\end{matrix}}{\theta}\ \text{RRA}$$

Def. A formula θ is said to be essentially negatable in L if there exists a formula $\theta^{\dot\sim}$ in L such that $\Gamma \models \theta \iff \Gamma, \theta^{\dot\sim} \models \bot$.

Theorem

Let $\Gamma \cup \{\theta\}$ be a set of formulas of L, where θ is essentially negatable in L. We have that $\Gamma \models \theta \iff \Gamma \vdash^*_{\mathsf{L}} \theta$.

Proof. Since the logic L is compact (Quadrellaro, Puljujärvi 2022), we may w.l.o.g. assume that Γ is finite. Then, we have that

$$\Gamma \models \theta \implies \Gamma, \theta^{\dot\sim} \models \bot$$, where $\dot\sim\theta$ stands for the L-formula $\phi \equiv \dot\sim\theta$

$$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \theta^{\dot\sim}) \models \bot$$, where $\vec{x}$ lists all free variables

$$\implies \exists\vec{x}(\bigwedge\Gamma \wedge \theta^{\dot\sim}) \vdash_L \bot \quad \text{(by the completeness thm)}$$

$$\implies \Gamma, \theta^{\dot\sim} \vdash_L \bot \quad \text{(by the rule } \exists\text{I)}$$

$$\implies \Gamma \vdash^*_L \theta. \quad \square$$

Example: Formulas θ in $\mathrm{FO}(\forall^1)$ and pseudo-flat sentences are ess. negatable in $\mathsf{L} = \mathrm{FO}(=\!(\ldots), \forall^1, \exists^1, \mathbin{\vee\!\!\!\vee})$ & $\theta^{\dot\sim} = \neg\theta^*$ (Kontinen, Y. 2022)

New rule:

$$\dfrac{\begin{matrix}[\theta^{\dot\sim}]\\ \vdots \\ \bot\end{matrix}}{\theta}\ \mathrm{RRA}$$

Def. A formula θ is said to be essentially negatable in L if there exists a formula $\theta^{\dot\sim}$ in L such that $\Gamma \models \theta \iff \Gamma, \theta^{\dot\sim} \models \bot$.

Another approach: finding weaker fully axiomatizable fragments

- (Baltag, van Benthem 2021, JPL)

  A "simple" logic for functional dependence

  – fully axiomatizable, and decidable

- (Kontinen, Y. 2022, JSL)

  A "first-order" version of team logic (FOT):

  $$\phi ::= R(\vec{t}) \mid t = t' \mid \vec{x} \subseteq \vec{y} \mid \dot{\sim} \phi \mid (\phi \wedge \phi) \mid (\phi \vee\!\!\!\vee \phi) \mid \exists^1 x\phi \mid \forall^1 x\phi$$

  where $\vec{x} \subseteq \vec{y}$ is the inclusion atom, $\vee\!\!\!\vee$ is the inquisitive disjunction, $\forall^1$ and $\exists^1$ are weaker quantifiers.

  – FOT $\equiv$ FO(R), and dependence and independence atoms are definable in FOT

  – fully axiomatizable (a complete deduction system given)