Towards Functorial Model Theory

LLAMA

February 8, 2023

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An Invitation

A famous quote from [Hodges, 1997]:

Model theory is algebraic geometry minus fields.

An Invitation

A longer quote from [Macintyre, 2003]:

> *I see model theory as becoming increasingly detached from set theory, and the Tarskian notion of set-theoretic model being no longer central to model theory. ... In algebraic geometry, schemes or algebraic spaces are the basic notions, with the older "sets of points in affined or projective space" no more than restrictive special cases. The basic notions may be given sheaf-theoretically, or functorially. ... The resulting relativization and "transfer of structure" is incomparably more flexible and powerful than anything yet known in "set-theoretic model theory".*

Modern algebraic geometry benefits a lot from sheaf theory (general cohomology) and category theory (base change techniques).

Question

- Can the same process also apply to model theory?
- Would it make the theory easier/more conceptual?
- Would this be anything that's potentially benefitial?

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Chaning a language is *way more harder* than learning techniques.

A lesson from physics:

$$Newton - - \succ Lagrange - - \succ Hamilton - - \succ QM$$

Best combo: *Simplicity*, *Beauty*, *Insight*, *Practicality* …

I'd like to discuss some preliminary observations, as an *invitation*.

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- Assume some CT and MT knowledge, but happy to explain.

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Outline

Syntactical to Categorical: A Bridge

Interpretations and Models: United

Finer Graded Invariance: Naturality

Conceptual Completeness: Duality

Some Wishful Thinking: Outlooks

Syntactical to Categorical: A Bridge

Theories as Categories

Slogan

Theories *are* categories (with some structures).

Similar to Lindenbaum-Tarski algebra, we can construct a *category* out of any first-order theory, extracting its *essential* information:

- Works for both classical and (various) more restricted theories.
  Benefits: view both $\mathbf{Grp}$ *and* $\mathbf{Grp}_{el}$ as categories of models.
- All/Most syntactic constructions can be made categorical.
  Benefits: *conceptual* arguments, rather than clever tricks.
- Compel oneself to consider (2-)categorical data.
  Benefits: necessary to have a *duality* for first-order theories.

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Syntactic Category of a Theory

Let $\mathbb{T}$ be a theory, construct a category $\mathcal{C}_{\mathbb{T}}$ as follows:

- Obj: α-equivalence classes of formulas $\varphi(x_1, \ldots, x_n)$.
- Mor: $\mathbb{T}$-equivalence classes of provably functional formulas $[\theta]$.

Some internal structures:

- For any $\varphi(\bar{x})$, $[\varphi] : \varphi(\bar{x}) \to \top$, which makes $\top$ terminal.
- For any object $\varphi(\bar{x})$, $\mathrm{Sub}(\varphi)$ consists of those $\psi(\bar{x})$ that

$$\mathbb{T} \vdash \forall \bar{x}(\psi(\bar{x}) \to \varphi(\bar{x})).$$

- For instance, $\mathrm{Sub}(\top)$ is the LT-algebra of sentences (e.g. $\mathbb{T}$ is complete iff $\mathcal{C}_{\mathbb{T}}$ is *two-valued*, $\mathrm{Sub}(\top) \cong \mathbf{2}$).

$\mathcal{C}_{\mathbb{T}}$ contains the syntactic and proof-theoretic information of $\mathbb{T}$.

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Categorical Structures of Syntactic Categories

In general, $\mathcal{C}_{\mathbb{T}}$ of a theory $\mathbb{T}$ will be a *coherent category*:

- It has all finite limits.
- It has (stable) image factorisation.
- For any φ, $\mathsf{Sub}(\varphi)$ will be a (stable) distributive lattice.

Example

- $\mathcal{C}_{PA}, \mathcal{C}_{ZF}$: Arithmetisation of meta-logic, inner models, could be viewed as constructions *internal* to these categories.
- **FinSet**, $\mathbf{FinSet}^{\mathbb{Z}}$,...
- $\mathbf{PER}(A)$ for any pca A.
- If $\mathcal{C}$ is coherent, then so is any slice $\mathcal{C}/\varphi$.
- Some large exmples: **Set**, **Comp**, and any topos.

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Interpretations and Models: United

Functorial Semantics

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Interpretations and models are the same as (certain) functors.

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A *model* of $\mathbb{T}$ in another coherent category $\mathcal{E}$ is simply a *coherent functor* from $\mathcal{C}_{\mathbb{T}}$ to $\mathcal{E}$. Semantic examples:

- A model in the usual sense is $M : \mathcal{C}_{\mathbb{T}} \to \mathbf{Set}$. In classical terms,

$$M(\varphi) = [\![\varphi]\!]_M = \{\, \overline{a} \in M^n \mid M \vDash \varphi[\overline{a}] \,\}.$$

- In **Comp**, a model is coherently equipped with a topology.
- In $\mathsf{Sh}(X)$, a model is a family continuously indexed by X.

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Interpretations and models are the same as (certain) functors.

Syntactically, a model $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{S}}$ is an *interpretation* of $\mathbb{T}$ in $\mathbb{S}$, in model theory usually denoted as $\mathbb{T} \leq \mathbb{S}$:

- If $\varphi \hookrightarrow \top$, then pulling back provides us

$$\mathcal{C}_{\mathbb{T}} \cong \mathcal{C}_{\mathbb{T}}/\top \to \mathcal{C}_{\mathbb{T}}/\varphi.$$

  This corresponds to the fact that $\mathbb{T} \leq \mathbb{T} + \varphi$.

Case Study I: Disjunctive Interpretation

A theorem in model theory: If $\tau : \mathbb{S} \leq \mathbb{T} + \varphi$ and $\sigma : \mathbb{S} \leq \mathbb{T} + \neg\varphi$, then one can construct *disjunctive interpretation* $\langle \tau, \sigma \rangle : \mathbb{S} \leq \mathbb{T}$:[1]

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Syntactically, the construction looks as follows,

- Domain: $\delta_{\langle \tau, \sigma \rangle}(x) := (\delta_\tau(x) \wedge \varphi) \vee (\delta_\sigma(x) \wedge \neg\varphi)$.
- For predicate P: $\psi_{P,\tau\times\sigma}(\overline{x}) := (\psi_{P,\tau}(\overline{x}) \wedge \varphi) \vee (\psi_{P,\sigma}(\overline{x}) \wedge \neg\varphi)$.
- For function f: ...

Then you need to verify that $\langle \tau, \sigma \rangle$ is a well-defined interpretation...

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Categorically you get this "for free" by abstract-nonsense:

$$\begin{array}{ccc} \bot & \longrightarrow & \varphi \\ \downarrow & & \downarrow \\ \neg\varphi & \longrightarrow & \top \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathbf{1} & \longleftarrow & \mathcal{C}_{\mathbb{T}}/\varphi \\ \uparrow & & \uparrow \\ \mathcal{C}_{\mathbb{T}}/\neg\varphi & \longleftarrow & \mathcal{C}_{\mathbb{T}} \end{array}$$

This gives $\mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}}/\neg\varphi \times \mathcal{C}_{\mathbb{T}}/\varphi$, hence $\langle \tau, \sigma \rangle$ is automatic.

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Case Study II: Elimination of Imaginaries

Given $\mathbb{T}$, we can syntactically construct $\mathbb{T}^{el}$, which "eliminates imaginaries", i.e. $\mathbb{T}^{el}$ realises all definable quotients in $\mathbb{T}$.

Categorically, this is expressed in the following reflexive adjunction,

$$\mathbf{Coh} \underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Pretopos}$$

This is an example where syntactical and categorical construction go hand in hand, but the latter makes the universal property explicit.

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Finer Graded Invariance: Naturality

Study Theories through Invariance

Just like algebraic topology and algebraic geometry, we study theories by associating them with some *invariance*.

The most common invariance is the *spectrum* of a theory: The number of (isomorphic copies of) models in each cardinality:

- Löwenheim-Skolem theorem, Vaught's test, Morley's categoricity theorem, Shelah's (and others) characterisation of the spectrum of a countable, complete theory, etc.

These things are powerful, but have their limitations.

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Homomorphisms between Models

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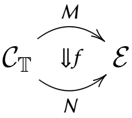
Model homomorphisms are the same as natural transformations.

A morphism between models is simply a natural transformation,



In particular, $\mathbf{Coh}(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$ is itself a category, not just a set. We use $\mathrm{Mod}(\mathcal{C}_{\mathbb{T}})$ to abbreviate $\mathbf{Coh}(\mathcal{C}_{\mathbb{T}}, \mathbf{Set})$.

This information is lost when only considering the model-theoretic spectrum, or only considering $\mathbb{S} \leq \mathbb{T}$ when studying interpretations.

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Case Study III: Classifying $\widehat{\mathbb{Z}}$-Actions

$\mathbf{FinSet}^{\mathbb{Z}}$ is the category of finite sets with $\mathbb{Z}$-action; equivalently, objects are (X, α) with X finite and $\alpha : X \to X$.

Consider $\mathbf{FinSet}^{\mathbb{Z}}$ as a theory: It is countable, classical, complete.

Theorem

$\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}}) \simeq B\widehat{\mathbb{Z}}$.

- $\widehat{\mathbb{Z}}$ has a single object, with automorphisms $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$.
- *Categorical*?! Attempts to write $\mathbf{FinSet}^{\mathbb{Z}}$ as a theory runs into problems (w.r.t. current definitional choices of model theory).
- More generally, for any group G, $\mathrm{Mod}(\mathbf{FinSet}^{G}) \simeq B\widehat{G}$.

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Conceptual Completeness: Duality

Let $\{M_s\}_{s\in S} \in \mathrm{Mod}(\mathcal{C}_{\mathbb{T}})^S$, and $\mu \in \beta S$. For any $A \subseteq S$, let

$$M^A := \prod_{s\in A} M_s.$$

Notice, if $B \subseteq A$, then we have a canonical projection $M^A \to M^B$. The ultraproduct M^μ is simply the following filtered colimit,

$$M^\mu := \varinjlim_{A\in\mu} M^A.$$

Theorem (Łos Ultraproduct theorem)

M^μ also lies in $\mathrm{Mod}(\mathcal{C}_{\mathbb{T}})$.

M^μ can be viewed as the following composition,

$$\mathcal{C}_{\mathbb{T}} \xrightarrow{M} \mathbf{Set}^S \xrightarrow{\int(-)d\mu} \mathbf{Set}$$

where $\int X_s d\mu = X^\mu$. Only need to verify it is coherent:

- M is because each M_s is, and (co)limits in $\mathbf{Set}^S$ are point-wise.
- $\int(-)d\mu$ is essentially because $\int(-)d\mu = \varinjlim_{A\in\mu}(-)^A$.

Ultracategories

[Makkai, 1987] introduced the notion of *ultracategories*, and it is further developed in [Lurie, 2018]:

Definition (Ultracategories)

An ultracategory is a category with an ultraproduct structure.

Example

- An *ultraset* is a compact Hausdorff space: **UltSet** $\cong$ **Comp**.
- $\mathrm{Mod}(D)$ for any distributive lattice D is an ultraposet.
- $\mathrm{Mod}(\mathcal{C}_{\mathbb{T}})$ for any coherent category $\mathcal{C}_{\mathbb{T}}$.
- The category of points of a topos $\mathcal{E}$ (locale L), if it is injective w.r.t. certain family of maps (cf. [Di Liberti, 2022]).
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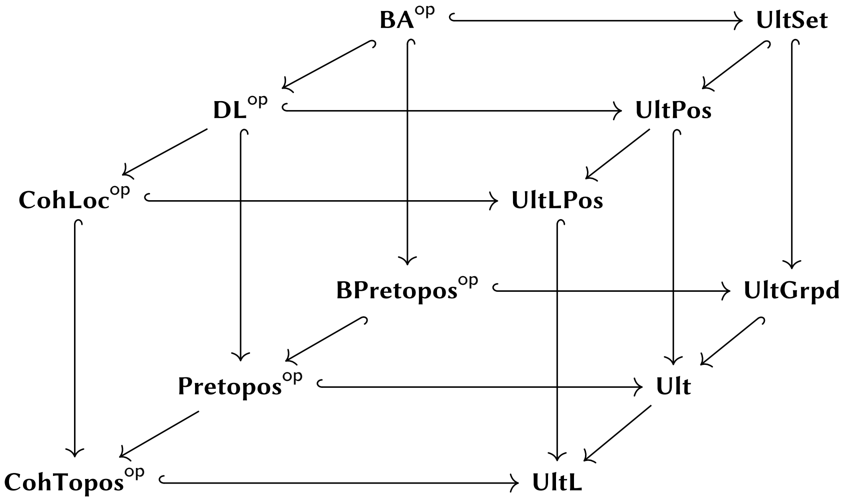
Duality of Logic

We have three 2-categories **UltL**, **UltR**, and **Ult** (cf. [Lurie, 2018]):



Case Study IV: Local Theories

Let $\mathcal{B}$ be a pretopos/first-order theory $\mathbb{T}$. It is *local*, if equivalently,

- **1** is connected and projective.
- $\mathbb{T}$ has the *disjunctive* and *existential* property.
- $\mathcal{B}(\mathbf{1}, -)$ is coherent (global section is a model).

Dualigy, $\mathcal{B}/\mathbb{T}$ is local iff $\mathrm{Mod}(\mathcal{B})$ is (roughly) $\overline{\{M\}}$ for some model M.

For any $\mathcal{B}/\mathbb{T}$ and any model M, $\mathcal{B}_M/\mathrm{Diag}(M)$ is local:

$$\begin{array}{ccc} \mathcal{B} & - - - - & \mathrm{Mod}(\mathcal{B}) \\ \downarrow & & \uparrow \\ \mathcal{B}_M & - - - & M/\mathrm{Mod}(\mathcal{B}) \end{array}$$

Completeness ≈ *presentation* with *local* theories, cf. [Awodey, 2021]:

$$\mathcal{B} \hookrightarrow \prod \mathcal{B}_M, \text{ or } \mathcal{B} \simeq \Gamma(\tilde{\mathcal{B}}) \text{ with } \tilde{\mathcal{B}} \text{ a sheaf of local pretoposes.}$$

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$$\begin{array}{ccc} \mathcal{B} & - - - - & \mathrm{Mod}(\mathcal{B}) \\ \downarrow & & \uparrow \\ \mathcal{B}_M & - - - & M/\mathrm{Mod}(\mathcal{B}) \end{array}$$

Completeness $\approx$ *presentation* with *local* theories, cf. [Awodey, 2021]:

$$\mathcal{B} \hookrightarrow \prod \mathcal{B}_M, \text{ or } \mathcal{B} \simeq \Gamma(\tilde{\mathcal{B}}) \text{ with } \tilde{\mathcal{B}} \text{ a sheaf of local pretoposes.}$$

Case Study V: Automorphism Groups of Models

Duality implies we have a correspondence:

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & - - - - & \mathrm{Mod}(\mathbb{T}) \\ \downarrow & & \uparrow \\ \mathbf{FinSet}^{\mathbb{Z}} & - - - - & B\widehat{\mathbb{Z}} \end{array}$$

Or equivalently, there is a bijection between

$$\left\{\text{models of } \mathbb{T} \text{ in } \mathbf{FinSet}^{\mathbb{Z}}\right\} \cong \left\{\mathbb{T}\text{-models with } \textit{closed}\ \widehat{\mathbb{Z}}\text{-action}\right\}$$

Corollary

No models of PA (ZF) has a closed $\widehat{\mathbb{Z}}$-action (or $\widehat{G}$-action).

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Some Wishful Thinking: Outlooks

In this talk, I try to (start to) convince:

- Model theory can be *beautifully*, *usefully* thought categorically.
- It leads to deeper understanding between theories and models.

Much more needs to be done:

- Serious development of model theory using *geometric* tools: vector bundles/quasi-coherent sheaves over space of models, (co)homology, fundamental group (at least in "tame" cases) ...
- Base change techniques/descent (cf. [Zawadowski, 1995]).
- Model theoretic Galois theory (cf. [Poizat, 1983])

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